Static Analysis of Power Systems

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Contents

Chapter 1

Introduction

1.1 The development of the Swedish power system

The Swedish power system started to develop around a number of hydro power stations, Porjus in Norrland, Älvkarleby in eastern Svealand, Motala in the middle of Svealand and Trollhättan in Götaland, at the time of the first world war. Later on, coal fired power plants located at larger cities as Stockholm, Göteborg, Malmö and Västerås came into operation. At the time for the second world war, a comprehensive proposal was made concerning exploitation of the rivers in the northern part of Sweden. To transmit this power to the middle and south parts of Sweden, where the heavy metal industry were located, a 220 kV transmission system was planned.

Today, the transmission system is well developed with a nominal voltage of 220 or 400 kV. In rough outline, the transmission system consists of lines, transformers and sub-stations.

A power plant can have an installed capacity of more than 1000 MW, e.g. the nuclear power plants Forsmark 3 and Oskarshamn 3, whereas an ordinary private consumer can have an electric power need of some kW. This implies that electric power can be generated at some few locations but the consumption, which shows large variations at single consumers, can be spread all over the country.

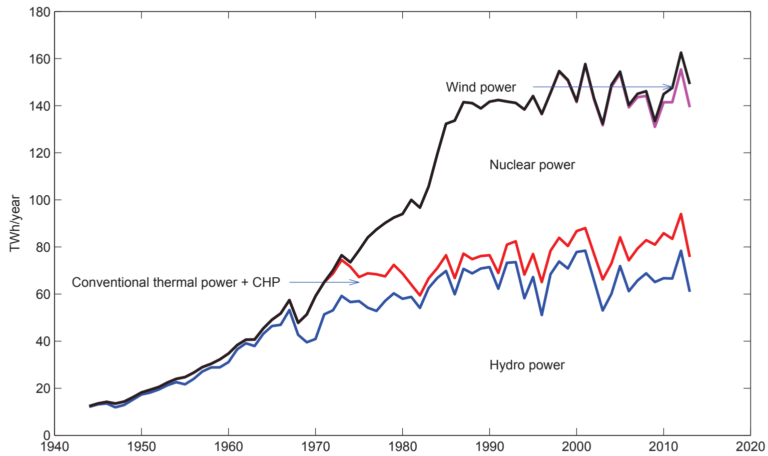


Figure 1.1. Electricity supply in Sweden 1944–2013

In Figure 1.1, the electricity supply in Sweden between 1944 and 2013 is given. The hydro power was in the beginning of this period the dominating source of electricity until the middle of the 1960s when some conventional thermal power plants (oil fired power plants, industrial back pressure, etc.) were taken into service. In the beginning of the 1970s, the first nuclear power plants were taken into operation and this power source has ever after being the one showing the largest increase in generated electric energy. Since around 1990, the trend showing a continuous high increase in electric power consumption has been broken.

In Table 1.1, the electricity supply in Sweden during 2014 is given.

Source of power	Energy generation TWh = 10^9 kWh	Installed capacity 14-12-31 MW
Hydro	64.2	16 155
Nuclear	62.2	9 528
Industrial back pressure	5.9	1 375
Combined heat and power	6.9	3 681
Oil fired condensing power	0.5	1 748
Gas turbine	0.01	1 563
Solar power	0.05	79
Wind power	11.5	54 20
Total	151.2	39 549

Table 1.1. Electricity supply in Sweden 2014

The total consumption of electricity is usually grouped into different categories. In Figure 1.2, the consumption from 1946-2013 is given for different groups.

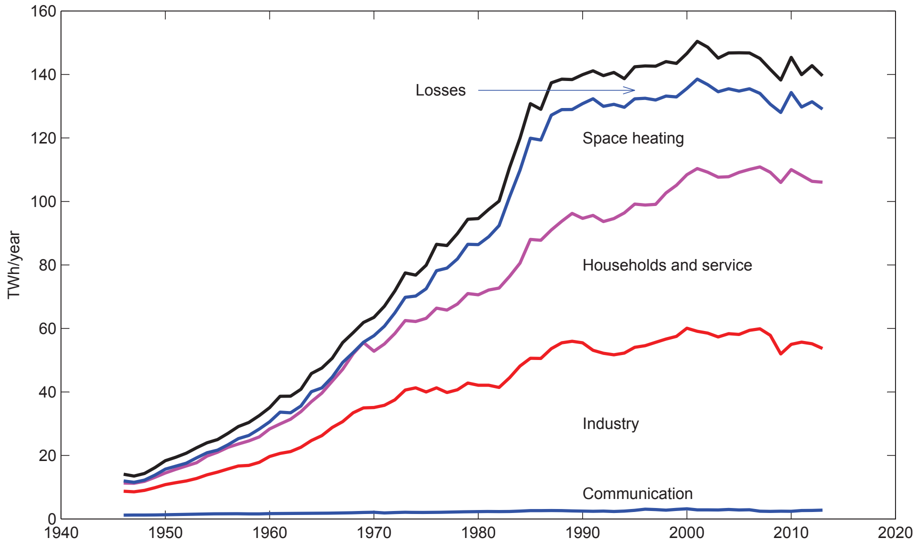


Figure 1.2. Consumption of electricity in Sweden 1946–2013

As shown in the figure, the major increase in energy need has earlier been dominated by the industry. When the nuclear power was introduced in the early 1970s, the electric space heating increased significantly. Before 1965, the electric space heating was only marginal. Communication, i.e. trains, trams and subway, has increased its consumption from 1.4 TWh/year in 1950 to 2.8 TWh/year in 2013.

In proportion to the total electricity consumption, the communication group has decreased from 8.5 % to 1.9 % during the same period. The losses on the transmission and distribution systems have during the period 1946–2013 decreased from around 14 % of total consumption to approximately 7.5 %.

1.2 The structure of the electric power system

A power system consists of generation sources which via power lines and transformers transmits the electric power to the end consumers.

The power system between the generation sources and end consumers is divided into different parts according to Figure 1.3.

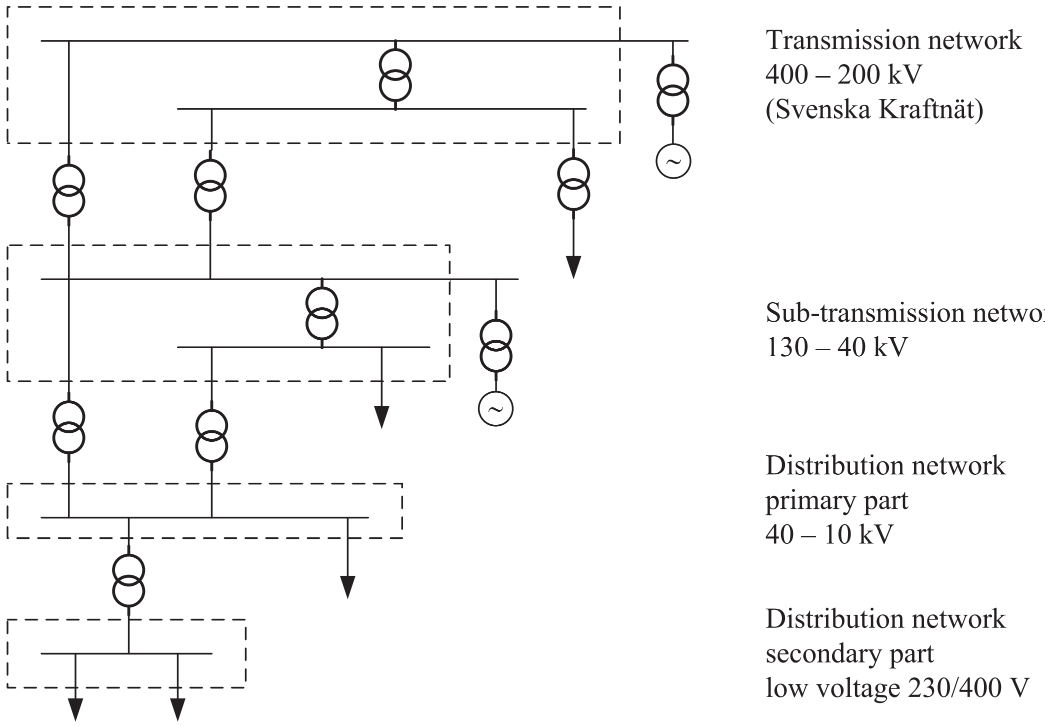


Figure 1.3. The structure of the electric power system

The transmission network, connects the main power sources and transmits a large amount of electric energy. The Swedish transmission system consists of approximately 15327 km power lines, and there are 16 interconnections to other countries. In Figure 1.4, a general map of the transmission system in Sweden and neighboring countries is given. The primary task for the transmission system is to transmit energy from generation areas to load areas. To achieve a high degree of efficiency and reliability, different aspects must be taken into account. The transmission system should for instance make it possible to optimize the generation within the country and also support trading with electricity with neighboring countries. It is also necessary to withstand different disturbances such as disconnection of transmission lines, lightning storms, outage of power plants as well as unexpected growth in power demand without reducing the quality of the electricity services. As shown in Figure 1.4, the transmission system is meshed, i.e. there are a number of closed loops in the transmission system.

A state utility, Svenska Kraftnät, manages the national transmission system and foreign links in operation at date. Svenska Kraftnät owns all 400 kV lines, all transformers between

Figure 1.4. Transmission system in north-western Europe

400 and 220 kV and the major part of the 220 kV lines in Sweden.

Sub-transmission network, in Sweden also called *regional network*, has in each load region the same or partly the same purpose as the transmission network. The amount of energy transmitted and the transmission distance are smaller compared with the transmission network which gives that technical-economical constraints implies lower system voltages. Regional networks are usually connected to the transmission network at two locations.

Distribution network, transmits and distributes the electric power that is taken from the sub-stations in the sub-transmission network and delivers it to the end users. The distribution network is in normal operation a radial network, i.e. there is only one path from the sub-transmission sub-station to the end user.

The electric power need of different end users varies a lot as well as the voltage level where the end user is connected. Generally, the higher power need the end user has, the higher voltage level is the user connected to.

The nominal voltage levels (Root Mean Square (RMS) value for tree-phase line-to-line (LL) voltages) used in distribution of *high voltage electric power* is normally lower compared with the voltage levels used in transmission. In Figure 1.5, the voltage levels used in Sweden are given. In special industry networks, except for levels given in Figure 1.5, also the voltage 660 V as well as the non-standard voltage 500 V are used. Distribution of *low voltage electric power* to end users is usually performed in three-phase lines with a zero conductor, which gives the voltage levels 400/230 V (line-to-line (LL)/line-to-neutral (LN) voltage).

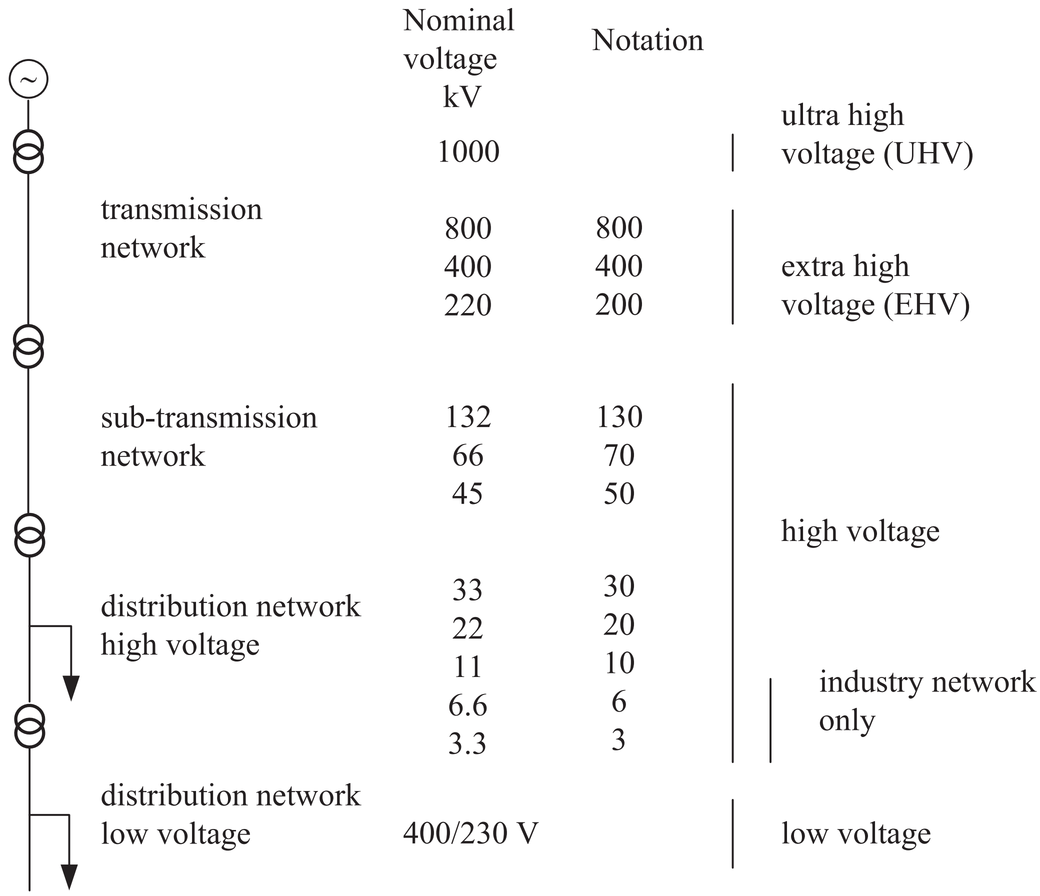


Figure 1.5. Standard voltage level for transmission and distribution. In Sweden, 400 kV is the maximum voltage

Chapter 2

Alternating current circuits

In this chapter, instantaneous and also complex power in an alternating current (AC) circuit is discussed. Also, the fundamental properties of AC voltage, current and power in a balanced (or symmetrical) three-phase circuit are presented.

2.1 Single-phase circuit

Assume that an AC voltage source with a sinusoidal voltage supplies a load as shown in Figure 2.1.

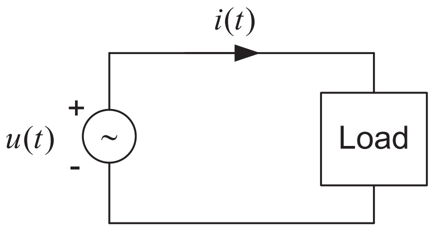


Figure 2.1. A sinusoidal voltage source supplies a load.

Let the instantaneous voltage and current be given by

$$\begin{aligned} u(t) &= U_M \cos(\omega t + \theta) \\ i(t) &= I_M \cos(\omega t + \gamma) \end{aligned} \tag{2.1}$$

where,

U_M	is	the peak value of the voltage,
I_M	is	the peak value of the current,
θ	is	the the phase angle of the voltage,
γ	is	the the phase angle of the current,
ω	$=$	$2\pi f$, and f is the frequency of the voltage source.

The single-phase instantaneous power consumed by the load is given by

$$\begin{aligned} p(t) &= u(t) \cdot i(t) = U_M I_M \cos(\omega t + \theta) \cos(\omega t + \gamma) = \\ &= \frac{1}{2} U_M I_M \left[\cos(\theta - \gamma) + \cos(2\omega t + \theta + \gamma)\right] = \\ &= \frac{U_M}{\sqrt{2}} \frac{I_M}{\sqrt{2}} [(1 + \cos(2\omega t + 2\theta)) \cos\phi + \sin(2\omega t + 2\theta) \sin\phi] = \\ &= P(1 + \cos(2\omega t + 2\theta)) + Q \sin(2\omega t + 2\theta) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\phi &= \theta - \gamma \\
P &= \frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}\cos\phi = U\,I\,\cos\phi = \text{active power} \\
Q &= \frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}\sin\phi = U\,I\,\sin\phi = \text{reactive power}
\end{aligned}$$

U and I are the Root Mean Square (RMS) value of the voltage and current, respectively. The RMS-values are defined as

$$U = \sqrt{\frac{1}{T}\int_0^T u(t)^2 dt} \tag{2.3}$$

$$I = \sqrt{\frac{1}{T}\int_0^T i(t)^2 dt} \tag{2.4}$$

With sinusoidal voltage and current, according to equation (2.1), the corresponding RMS-values are given by

$$U = \sqrt{\frac{1}{T}\int_0^T U_M^2\cos^2(\omega t+\theta)} = U_M\sqrt{\frac{1}{T}\int_0^T\left(\frac{1}{2}+\frac{\cos(2\omega t+2\theta)}{2}\right)} = \frac{U_M}{\sqrt{2}} \tag{2.5}$$

$$I = \sqrt{\frac{1}{T}\int_0^T I_M^2\cos^2(\omega t+\gamma)} = \frac{I_M}{\sqrt{2}} \tag{2.6}$$

As shown in equation (2.2), the instantaneous power has been decomposed into two components. The first component has a *mean value* P, and pulsates with the double frequency. The second component also pulsates with double frequency with a *amplitude* Q, but it has a zero mean value. In Figure 2.2, the instantaneous voltage, current and power are shown.

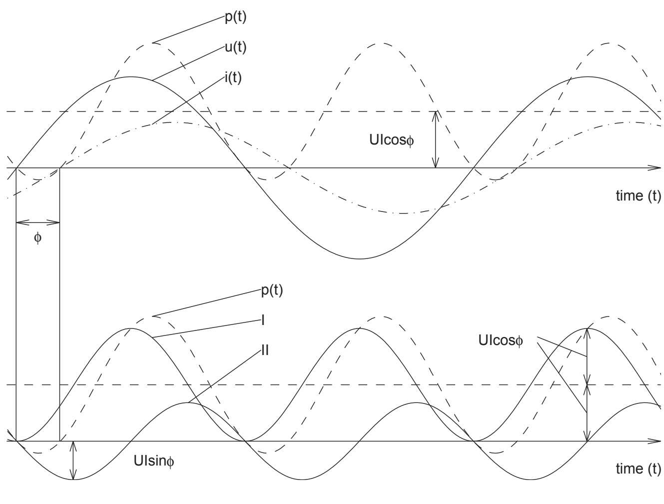


Figure 2.2. Voltage, current and power versus time.

Example 2.1 *A resistor of 1210 Ω is fed by an AC voltage source with frequency 50 Hz and voltage 220 V (RMS). Find the mean value power (i.e. the active power) consumed by the resistor.*

Solution

The consumed mean value power over one period can be calculated as

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{1}{T}\int_0^T R\cdot i^2(t)dt = \frac{1}{T}\int_0^T R\frac{u^2(t)}{R^2}dt = \frac{1}{R}\frac{1}{T}\int_0^T u^2(t)dt$$

which can be rewritten according to equation (2.3) as

$$P = \frac{1}{R}U^2 = \frac{220^2}{1210} = 40 \text{ W}$$

2.1.1 Complex power

The complex method is a powerful tool for calculation of electrical power, and can offer solutions in an elegant manner.

The single-phase phasor voltage and current are expressed by

$$\begin{aligned} \overline{U} &= Ue^{j\theta} \\ \overline{I} &= Ie^{j\gamma} \end{aligned} \tag{2.7}$$

where, U is the magnitude (RMS-value) of the voltage phasor, and θ is its phase angle. Also, I is the magnitude (RMS-value) of the current phasor, and γ is its phase angle.

The complex power ($\overline{S}$) is expressed by

$$\overline{S} = Se^{j\phi} = P + jQ = \overline{U}\,\overline{I}^* = UIe^{j(\theta-\gamma)} = UIe^{j\phi} = UI(\cos\phi + j\sin\phi) \tag{2.8}$$

which implies that

$$\begin{aligned} P &= S\cos\phi = UI\cos\phi \\ Q &= S\sin\phi = UI\sin\phi \end{aligned} \tag{2.9}$$

where, P is called active power, Q is called reactive power and $\cos\phi$ is called power factor.

Example 2.2 *Calculate the complex power consumed by an inductor with the inductance of 3.85 H which is fed by an AC voltage source with the phasor* $\overline{U} = U\angle\theta = 220\angle 0$ *V. The circuit frequency is 50 Hz.*

Solution

The impedance is given by

$$\overline{Z} = j\omega L = j \cdot 2 \cdot \pi \cdot 50 \cdot 3.85 = j1210 \ \Omega$$

Next, the phasor current through the impedance can be calculated as

$$\overline{I} = \frac{\overline{U}}{\overline{Z}} = \frac{220}{j1210} = -j0.1818 \ \text{A} = 0.1818\, e^{-j\frac{\pi}{2}} \ \text{A}$$

Thus, the complex power is given by

$$\begin{aligned} \overline{S} &= \overline{U}\,\overline{I}^* = UIe^{j(\theta-\gamma)} = UIe^{j(\phi)} \\ &= 220\,(0.1818)\,e^{j(0+\frac{\pi}{2})} = 220(j0.1818) = j40 \ \text{VA} \end{aligned}$$

i.e. P = 0 W, Q = 40 VAr.

Example 2.3 *Two series connected impedances are fed by an AC voltage source with the phasor $\overline{U}_1 = 1\angle 0$ V as shown in Figure 2.3.*

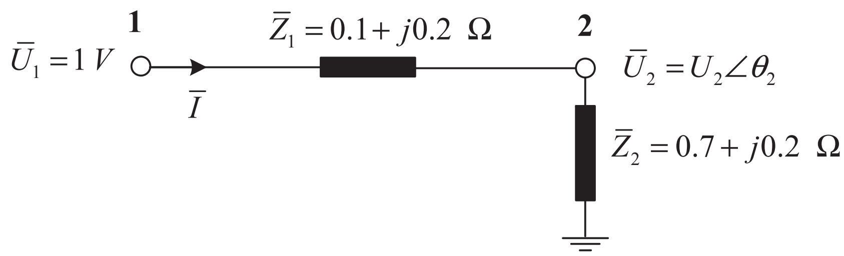


Figure 2.3. Network used in Example 2.3.

a) Calculate the power consumed by $\overline{Z}_2$ as well as the power factor ($\cos\phi$) at bus 1 and 2 where ϕ_k is the phase angle between the voltage and the current at bus k.

b) Calculate the magnitude U_2 when $\overline{Z}_2$ is capacitive : $\overline{Z}_2 = 0.7 - j0.5\ \Omega$

Solution

a)

$$\overline{U}_1 = U_1\angle\theta_1 = 1\angle 0 \ \text{V} \quad \text{and} \quad \overline{I} = \frac{\overline{U}_1}{\overline{Z}_1 + \overline{Z}_2} = I\angle\gamma = 1.118\angle - 26.57^\circ \ \text{A}$$

Thus, $\phi_1 = \theta_1 - \gamma = 26.57^\circ$, and $\cos\phi_1 = 0.8944$ lagging, since the current lags the voltage. Furthermore,

$$\overline{U}_2 = \overline{Z}_2 \cdot \overline{I} = U_2\angle\theta_2 = 0.814\angle - 10.62^\circ$$

Thus, $\phi_2 = \theta_2 - \gamma = -10.62^\circ + 26.57^\circ = 15.95^\circ$, and $\cos\phi_2 = 0.9615$, lagging. The equation above can be written on polar form as

$$\begin{aligned} U_2 &= Z_2 \cdot I \\ \theta_2 &= \arg(\overline{Z}_2) + \gamma \end{aligned}$$

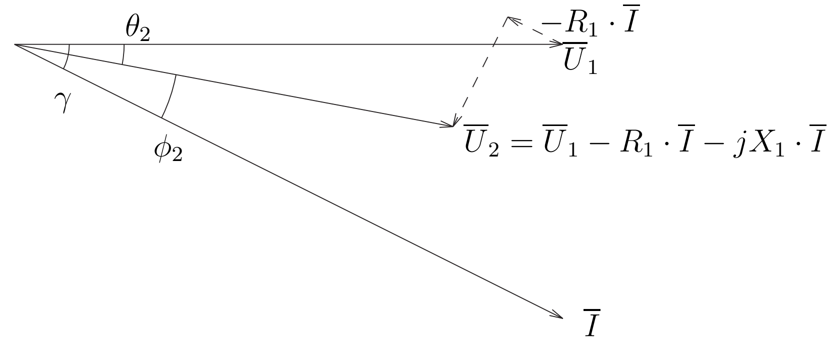


Figure 2.4. Solution to Example 2.3 a).

i.e. $\phi_2 = \arg(\overline{Z}_2) = \arctan\frac{X_2}{R_2} = 15.95°$

The power consumption in $\overline{Z}_2$ can be calculated as

$$\overline{S}_2 = P_2 + jQ_2 = \overline{Z}_2 \cdot I^2 = (0.7 + j0.2)1.118^2 = 0.875 + j0.25 \text{ VA}$$

or

$$\overline{S}_2 = P_2 + jQ_2 = \overline{U}_2\overline{I}^* = U_2\, I\, \angle\phi_2 = 0.814 \cdot 1.118\angle 15.95° = 0.875 + j0.25 \text{ VA}$$

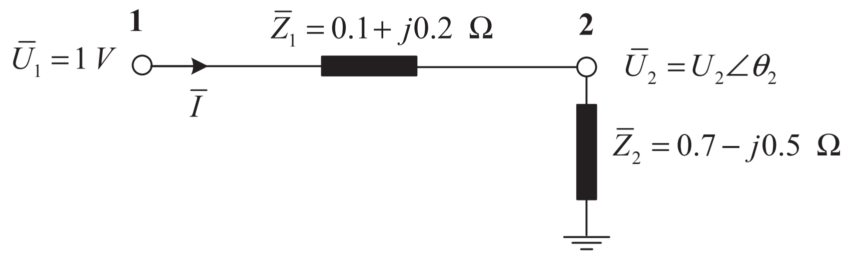


Figure 2.5. Network used in Example 2.3 b).

b)

$$U_2 = \left|\frac{\overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}\right| U_1 = \frac{|0.7 - j0.5|}{|0.8 - j0.3|} = \frac{\sqrt{0.49 + 0.25}}{\sqrt{0.64 + 0.09}} = \frac{\sqrt{0.74}}{\sqrt{0.73}} = 1.007 \text{ V}$$

Conclusions from this example are that

- a capacitance increases the voltage - so called phase compensation,
- active power can be transmitted towards higher voltage magnitude,
- the power factor $\cos\phi$ may be different in different ends of a line,
- the line impedances are $\ll$ load impedances.

2.2 Balanced three-phase circuit

In a balanced (or symmetrical) three-phase circuit, a three-phase voltage source consists of three AC voltage sources (one for each phase) which are equal in amplitude (or magnitude) and displaced in phase by 120°. Furthermore, each phase is equally loaded.

Let the instantaneous phase (also termed as line-to-neutral (LN)) voltages be given by

$$\begin{aligned} u_a(t) &= U_M \cos(\omega t + \theta) \\ u_b(t) &= U_M \cos(\omega t + \theta - \frac{2\pi}{3}) \\ u_c(t) &= U_M \cos(\omega t + \theta + \frac{2\pi}{3}) \end{aligned} \tag{2.10}$$

Variations of the three voltages versus time are shown in Figure 2.6.

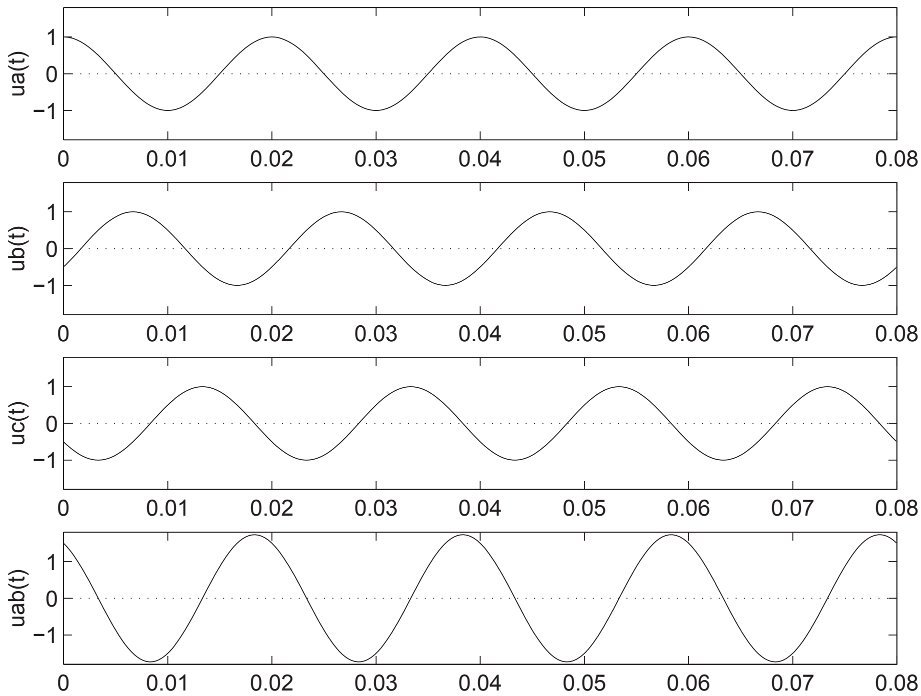


Figure 2.6. $u_a(t)$, $u_b(t)$, $u_c(t)$ and $u_{ab}(t)$ versus time with f = 50 Hz, $U_M = 1$ and $\theta = 0$.

For analysis of a balanced three-phase system, it is very common to use the voltage between two phases. This voltage is termed as *line-to-line (LL) voltage*. The line-to-line voltage u_{ab} is given by

$$\begin{aligned} u_{ab}(t) &= u_a(t) - u_b(t) = U_M \cos(\omega t + \theta) - U_M \cos(\omega t + \theta - \frac{2\pi}{3}) = \\ &= \sqrt{3}\, U_M \cos(\omega t + \theta + \frac{\pi}{6}) \end{aligned} \tag{2.11}$$

As shown in equation (2.11), in a balanced three-phase circuit the line-to-line voltage leads the line-to-neutral voltage by 30°, and is $\sqrt{3}$ times larger in amplitude (or magnitude, see equation (2.5)). For instance, at a three-phase power outlet the magnitude of a phase is 230 V, but the magnitude of a line-to-line voltage is $\sqrt{3}\cdot 230 = 400$ V, i.e. $U_{LL} = \sqrt{3}\,U_{LN}$. The line-to-line voltage u_{ab} is shown at the bottom of Figure 2.6.

Next, assume that the voltages given in equation (2.10) supply a balanced (or symmetrical) three-phase load whose phase currents are

$$\begin{aligned} i_a(t) &= I_M\cos(\omega t+\gamma) \\ i_b(t) &= I_M\cos(\omega t+\gamma-\frac{2\pi}{3}) \\ i_c(t) &= I_M\cos(\omega t+\gamma+\frac{2\pi}{3}) \end{aligned} \tag{2.12}$$

Then, the total instantaneous power is given by

$$\begin{aligned} p_3(t) &= p_a(t)+p_b(t)+p_c(t) = u_a(t)i_a(t)+u_b(t)i_b(t)+u_c(t)i_c(t) = \\ &= \frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}[(1+\cos 2(\omega t+\theta))\cos\phi+\sin 2(\omega t+\theta)\sin\phi]+ \\ &+ \frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}[(1+\cos 2(\omega t+\theta-\frac{2\pi}{3}))\cos\phi+\sin 2(\omega t+\theta-\frac{2\pi}{3})\sin\phi]+ \\ &+ \frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}[(1+\cos 2(\omega t+\theta+\frac{2\pi}{3}))\cos\phi+\sin 2(\omega t+\theta+\frac{2\pi}{3})\sin\phi] = \\ &= 3\frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}\Big[\cos\phi+\underbrace{\Big(\cos 2(\omega t+\theta)+\cos 2[\omega t+\theta-\frac{2\pi}{3}]+\cos 2[\omega t+\theta+\frac{2\pi}{3}]\Big)}_{=0}+ \\ &+ \underbrace{\Big(\sin 2(\omega t+\theta)+\sin 2[\omega t+\theta-\frac{2\pi}{3}]+\sin 2[\omega t+\theta+\frac{2\pi}{3}]\Big)}_{=0}\Big] = \\ &= 3\,\frac{U_M}{\sqrt{2}}\frac{I_M}{\sqrt{2}}\cos\phi = 3\,U_{LN}\,I\,\cos\phi \end{aligned} \tag{2.13}$$

Note that the total instantaneous power is equal to three times the active power of a single phase, and it is constant. This is one of the main reasons why three-phase systems have been used.

2.2.1 Complex power

The corresponding phasor voltages are defined as:

$$\begin{aligned} \overline{U}_a &= U_{LN}\angle\,\theta \\ \overline{U}_b &= U_{LN}\angle\,(\theta-120^\circ) \\ \overline{U}_c &= U_{LN}\angle\,(\theta+120^\circ) \end{aligned} \tag{2.14}$$

Figure 2.7 shows the phasor diagram of the three balanced line-to-neutral voltages, and also the phasor diagram of the line-to-line voltages.

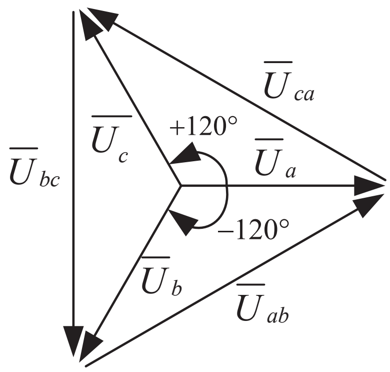


Figure 2.7. Phasor diagram of the line-to-neutral and line-to-line voltages.

The phasor of the line-to-line voltages can be determined as follows

$$\begin{aligned}
\overline{U}_{ab} &= \overline{U}_a - \overline{U}_b = \sqrt{3}\,U_{LN}\angle\,(\theta + 30°) &= \sqrt{3}\,\overline{U}_a\,e^{j30°} \\
\overline{U}_{bc} &= \overline{U}_b - \overline{U}_c = \sqrt{3}\,U_{LN}\angle\,(\theta - 90°) &= \sqrt{3}\,\overline{U}_b\,e^{j30°} \\
\overline{U}_{ca} &= \overline{U}_c - \overline{U}_a = \sqrt{3}\,U_{LN}\angle\,(\theta + 150°) &= \sqrt{3}\,\overline{U}_c\,e^{j30°}
\end{aligned} \tag{2.15}$$

Obviously, the line-to-line voltages are also balanced. Equation (2.15) also shows that the line-to-line phasor voltage leads the line-to-neutral phasor voltage by 30°, and it is $\sqrt{3}$ times the line-to-neutral phasor voltage.

Next, let the balanced phasor currents be defined as

$$\begin{aligned}
\overline{I}_a &= I\angle\,\gamma \\
\overline{I}_b &= I\angle\,(\gamma - 120°) \\
\overline{I}_c &= I\angle\,(\gamma + 120°)
\end{aligned} \tag{2.16}$$

Then, the total three-phase power ($\overline{S}_{3\Phi}$) is given by :

$$\begin{aligned}
\overline{S}_{3\Phi} &= \overline{S}_a + \overline{S}_b + \overline{S}_c = \overline{U}_a\overline{I}_a^* + \overline{U}_b\overline{I}_b^* + \overline{U}_c\overline{I}_c^* = \\
&= 3\,U_{LN}\,I\,\cos\phi + j3\,U_{LN}\,I\,\sin\phi = \\
&= 3\,U_{LN}\,I\,e^{j\phi}
\end{aligned} \tag{2.17}$$

Obviously, for a balanced three-phase system $\overline{S}_a = \overline{S}_b = \overline{S}_c$ and $\overline{S}_{3\Phi} = 3\,\overline{S}_{1\Phi}$, where $\overline{S}_{1\Phi}$ is the complex power of a single phase.

Example 2.4 *The student Elektra lives in a house situated 2 km from a transformer having a completely symmetrical three-phase voltage ($\overline{U}_a = 220V\angle 0°, \overline{U}_b = 220V\angle -120°, \overline{U}_c = 220V\angle 120°$). The house is connected to this transformer via a three-phase cable (EKKJ, 3×16 mm^2 + 16 mm^2). A cold day, Elektra switches on two electrical radiators to each phase, each radiator is rated 1000 W (at 220 V with cosϕ = 0.995 lagging (inductive)). Assume that the cable can be modeled as four impedances connected in parallel ($\overline{z}_L$ = 1.15 + j0.08 Ω/phase,km, $\overline{z}_{L0}$ = 1.15 + j0.015 Ω/km) and that the radiators also can be considered as impedances. Calculate the total thermal power given by the radiators.*

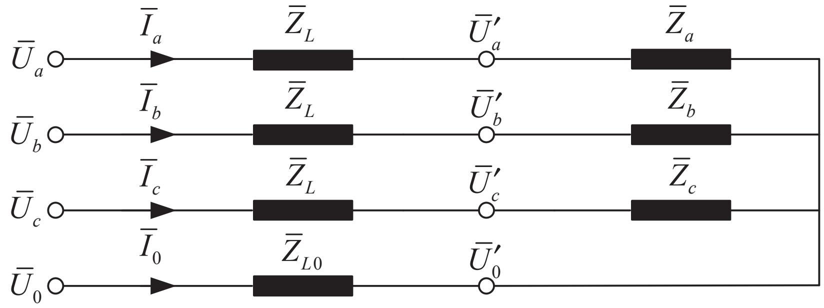


Figure 2.8. The network diagram in Example 2.4.

Solution

$\overline{U}_a = 220\angle 0°$ V, $\overline{U}_b = 220\angle -120°$ V, $\overline{U}_c = 220\angle 120°$ V

$\overline{Z}_L = 2(1.15 + j0.08) = 2.3 + j0.16\ \Omega$

$\overline{Z}_{L0} = 2(1.15 + j0.015) = 2.3 + j0.03\ \Omega$

$P_a = P_b = P_c = 2000$ W (at 220 V, $\cos\phi = 0.995$)

$\sin\phi = \sqrt{1-\cos^2\phi} = 0.0999$

$Q_a = Q_b = Q_c = S\sin\phi = \dfrac{P}{\cos\phi}\sin\phi = 200.8$ VAr

$\overline{Z}_a = \overline{Z}_b = \overline{Z}_c = \frac{\overline{U}}{\overline{I}} = \frac{\overline{U}\cdot\overline{U}^*}{\overline{I}\cdot\overline{U}^*} = U^2/\overline{S}^* = U^2/(P_a - jQ_a) = 23.96 + j2.40\ \Omega$

$\overline{I}_a = \frac{\overline{U}_a - \overline{U}'_0}{\overline{Z}_L + \overline{Z}_a} \qquad \overline{I}_b = \frac{\overline{U}_b - \overline{U}'_0}{\overline{Z}_L + \overline{Z}_b} \qquad \overline{I}_c = \frac{\overline{U}_c - \overline{U}'_0}{\overline{Z}_L + \overline{Z}_c}$

$\overline{I}_a + \overline{I}_b + \overline{I}_c = \frac{\overline{U}'_0 - \overline{U}_0}{\overline{Z}_{L0}} = \frac{\overline{U}'_0 - 0}{\overline{Z}_{L0}}$

$\Rightarrow \overline{U}'_0\left[\frac{1}{\overline{Z}_{L0}} + \frac{1}{\overline{Z}_L+\overline{Z}_a} + \frac{1}{\overline{Z}_L+\overline{Z}_b} + \frac{1}{\overline{Z}_L+\overline{Z}_c}\right] = \frac{\overline{U}_a}{\overline{Z}_L+\overline{Z}_a} + \frac{\overline{U}_b}{\overline{Z}_L+\overline{Z}_b} + \frac{\overline{U}_c}{\overline{Z}_L+\overline{Z}_c}$

$\Rightarrow \overline{U}'_0 = 0.0$

$\Rightarrow \overline{I}_a = 8.34\angle -5.58°$ A, $\overline{I}_b = 8.34\angle -125.58°$ A, $\overline{I}_c = 8.34\angle 114.42°$ A

The voltage at the radiators can be calculated as :

$\overline{U}'_a = \overline{U}'_0 + \overline{I}_a\overline{Z}_a = 200.78\angle 0.15°$ V

$\overline{U}_b = 200.78\angle -119.85°$ V

$\overline{U}_c = 200.78\angle 120.15°$ V

Finally, the power to the radiators can be calculated as

$\overline{S}_{za} = \overline{Z}_a I_a^2 = 1666 + j167$ VA

$\overline{S}_{zb} = \overline{Z}_b I_b^2 = 1666 + j167$ VA

$\overline{S}_{zc} = \overline{Z}_a I_c^2 = 1666 + j167$ VA

Thus, the total consumed power is

$\overline{S}_{za} + \overline{S}_{zb} + \overline{S}_{zc} = 4998 + j502$ VA, i.e. the thermal power $= 4998$ W

Note that since we are dealing with a balanced three-phase system, $\overline{S}_{za} = \overline{S}_{zb} = \overline{S}_{zc}$.

The total transmission losses are

$\overline{Z}_L(I_a^2 + I_b^2 + I_c^2) + \overline{Z}_{L0}|\overline{I}_a + \overline{I}_b + \overline{I}_c|^2) = \overline{Z}_L(I_a^2 + I_b^2 + I_c^2) = 480 + j33$ VA

i.e. the active losses are 480 W, which means that the efficiency is 91.2 %.

In a balanced three-phase system, $\overline{I}_a + \overline{I}_b + \overline{I}_c = 0$. Thus, no current flows in the neutral conductor (i.e. $\overline{I}_0 = 0$), and the voltage at the neutral point is zero, i.e. $\overline{U}_0' = 0$. Therefore, for analyzing a balanced three-phase system, it is more common to analyze only a single phase (or more precisely only the positive-sequence network of the system, see Chapter 8). Then, the total three-phase power can be determined as three times the power of the single phase.

Example 2.5 *Use the data in Example 2.4, but in this example the student Elektra connects one 1000 W radiator (at 220 V with cosϕ = 0.995 lagging) to phase a, three radiators to phase b and two to phase c. Calculate the total thermal power given by the radiators, as well as the system losses.*

Solution

$\overline{U}_a = 220\angle 0°\ \text{V}, \overline{U}_b = 220\angle -120°\ \text{V}, \overline{U}_c = 220\angle 120°\ \text{V}$

$\overline{Z}_L = 2(1.15 + j0.08) = 2.3 + j0.16\ \Omega$

$\overline{Z}_{L0} = 2(1.15 + j0.015) = 2.3 + j0.03\ \Omega$

$P_a = 1000$ W (at 220 V, $\cos\phi = 0.995$)

$\sin\phi = \sqrt{1 - \cos^2\phi} = 0.0999$

$Q_a = S\sin\phi = \dfrac{P}{\cos\phi}\sin\phi = 100.4$ VAr

$\overline{Z}_a = U^2/\overline{S}_a^* = U^2/(P_a - jQ_a) = 47.9 + j4.81\ \Omega$

$\overline{Z}_b = \overline{Z}_a/3 = 15.97 + j1.60\ \Omega$

$\overline{Z}_c = \overline{Z}_a/2 = 23.96 + j2.40\ \Omega$

$\overline{U}_0' \left[\frac{1}{\overline{Z}_{L0}} + \frac{1}{\overline{Z}_L + \overline{Z}_a} + \frac{1}{\overline{Z}_L + \overline{Z}_b} + \frac{1}{\overline{Z}_L + \overline{Z}_c}\right] = \frac{\overline{U}_a}{\overline{Z}_L + \overline{Z}_a} + \frac{\overline{U}_b}{\overline{Z}_L + \overline{Z}_b} + \frac{\overline{U}_c}{\overline{Z}_L + \overline{Z}_c}$

$\Rightarrow \overline{U}_0' = 12.08\angle -155.14°$ V

$\Rightarrow \overline{I}_a = 4.58\angle -4.39°\ \text{A}, \overline{I}_b = 11.45\angle -123.62°\ \text{A}, \overline{I}_c = 8.31\angle 111.28°$ A

The voltages at the radiators can be calculated as :

$\overline{U}_a' = \overline{U}_0' + \overline{I}_a\overline{Z}_a = 209.45\angle 0.02°$ V

$\overline{U}_b' = \overline{U}_0' + \overline{I}_b\overline{Z}_b = 193.60\angle -120.05°$ V

$\overline{U}_c' = \overline{U}_0' + \overline{I}_c\overline{Z}_c = 200.91\angle 129.45°$ V

Note that these voltages are not local phase voltages since they are calculated as $\overline{U}_a' - \overline{U}_0'$ etc. The power to the radiators can be calculated as :

$\overline{S}_{za} = \overline{Z}_a I_a^2 = 1004 + j101$ VA

$\overline{S}_{zb} = \overline{Z}_b I_b^2 = 2095 + j210$ VA

$\overline{S}_{zc} = \overline{Z}_a I_c^2 = 1655 + j166$ VA

The total amount of power consumed is

$\overline{S}_{za} + \overline{S}_{zb} + \overline{S}_{zc} = 4754 + j477$ VA, i.e. the thermal power is 4754 W

The total transmission losses are

$\overline{Z}_L(I_a^2 + I_b^2 + I_c^2) + \overline{Z}_{L0}|\overline{I}_a + \overline{I}_b + \overline{I}_c|^2) = 572.1 + j36$ VA, i.e. 572.1 W

which gives an efficiency of 89.3 %.

As shown in this example, an unsymmetrical impedance load will result in unsymmetrical phase currents, i.e. we are dealing with an unbalanced three-phase system. As a consequence, a voltage can be detected at the neutral point (i.e. $\overline{U}_0' \neq 0$) which gives rise to a current in the neutral conductor, i.e. $\overline{I}_0 \neq 0$. The total thermal power obtained was reduced by approximately 5 % and the line losses increased partly due to the losses in the neutral conductor. The efficiency of the transmission decreased. It can also be noted that the power per radiator decreased with the number of radiators connected to the same phase. This owing to the fact that the voltage at the neutral point will be closest to the voltage in the phase with the lowest impedance, i.e. the phase with the largest number of radiators connected.

Chapter 3
Models of power system components

Electric energy is transmitted from power plants to consumers via overhead lines, cables and transformers. In the following, these components will be discussed and mathematical models to be used in the analysis of symmetrical three-phase systems will be derived. In Chapter 8, analysis of power systems under unsymmetrical conditions will be discussed.

3.1 Electrical characteristic of an overhead line

Overhead transmission lines need large surface area and are mostly suitable to be used in rural areas and in areas with low population density. In areas with high population density and urban areas cables are a better alternative. For a certain amount of power transmitted, a cable transmission system is about ten times as expensive as an overhead transmission system.

Power lines have a *resistance* (r) owing to the resistivity of the conductor and a *shunt conductance* (g) because of leakage currents in the insulation. The lines also have an *inductance* (l) owing to the magnetic flux surrounding the line as well as a *shunt capacitance* (c) because of the electric field between the lines and between the lines and ground. These quantities are given per unit length and are continuously distributed along the whole length of the line. Resistance and inductance are in series while the conductance and capacitance are shunt quantities.

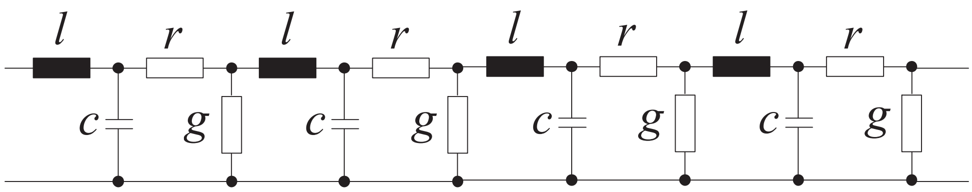


Figure 3.1. A line with distributed quantities.

Assuming symmetrical three-phase, a line can be modeled as shown in Figure 3.1. The quantities r, g, l, and c determine the characteristics of a line. Power lines can be modeled by simple equivalent circuits which, together with models of other system components, can be formed to a model of a complete system or parts of it. This is important since such models are used in power system analysis where active and reactive power flows in the network, voltage levels, losses, power system stability and other properties at disturbances as e.g. short circuits, are of interest.

For a more detailed derivation of the expressions of inductance and capacitance given below, more fundamental literature in electro-magnetic theory has to be studied.

3.1.1 Resistance

The resistance of a conductor with the cross-section area A mm^2 and the resistivity ρ $\Omega\, mm^2$/km is

$$r = \frac{\rho}{A} \quad \Omega/\text{km} \tag{3.1}$$

The conductor is made of copper with the resistivity at $20°C$ of 17.2 $\Omega\, mm^2$/km, or aluminum with the resistivity at $20°C$ of 27.0 $\Omega\, mm^2$/km. The choice between copper or aluminum is related to the price difference between the materials.

The effective alternating current resistance at normal system frequency (50–60 Hz) for lines with a small cross-section area is close to the value for the direct current resistance. For larger cross-section areas, the current density will not be equal over the whole cross-section. The current density will be higher at the peripheral parts of the conductor. That phenomena is called *current displacement* or *skin effect* and depends on the internal magnetic flux of the conductor. The current paths that are located in the center of the conductor will be surrounded by the whole internal magnetic flux and will consequently have an internal self inductance. Current paths that are more peripheral will be surrounded by a smaller magnetic flux and thereby have a smaller internal inductance.

The resistance of a line is given by the manufacturer where the influence of the skin effect is taken. Normal values of the resistance of lines are in the range 10–0.01 Ω/km.

The resistance plays, compared with the reactance, often a minor role when comparing the transmission capability and voltage drop between different lines. For low voltage lines and when calculating the losses, the resistance is of significant importance.

3.1.2 Shunt conductance

The shunt conductance of an overhead line represents the losses owing to leakage currents at the insulators. There are no reliable data over the shunt conductances of lines and these are very much dependent on humidity, salt content and pollution in the surrounding air. For cables, the shunt conductance represents the dielectric losses in the insulation material and data can be obtained from the manufacturer.

The dielectric losses are e.g. for a 12 kV cross-linked polyethylene (XLPE) cable with a cross-section area of 240 mm^2/phase 7 W/km,phase and for a 170 kV XLPE cable with the same area 305 W/km,phase.

The shunt conductance will be neglected in all calculations throughout this compendium.

3.1.3 Inductance

The inductance is in most cases the most important parameter of a line. It has a large influence on the line transmission capability, voltage drop and indirectly the line losses. The inductance of a line can be calculated by the following formula :

$$l = 2 \cdot 10^{-4} \left(\ln \frac{a}{d/2} + \frac{1}{4n} \right) \text{ H/km,phase} \tag{3.2}$$

where

$a \;=\; \sqrt[3]{a_{12}a_{13}a_{23}}$ m, = geometrical mean distance according to Figure 3.2.

d = diameter of the conductor, m

n = number of conductors per phase

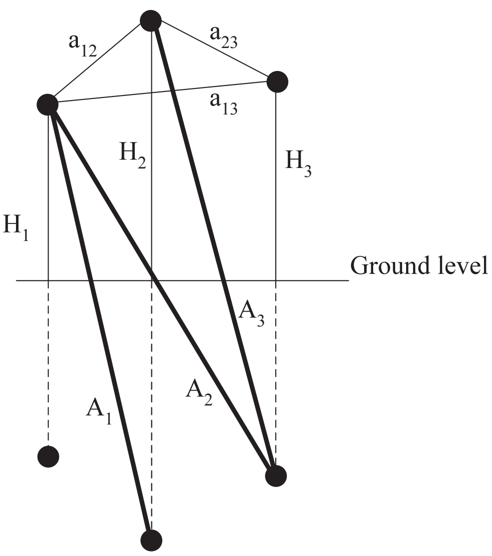


Figure 3.2. The geometrical quantities of a line in calculations of inductance and capacitance.

The calculation of the inductance according to equation (3.2), is made under some assumptions, viz. the conductor material must be non-magnetic as copper and aluminum together with the assumption that the line is transposed. The majority of the long transmission lines are transposed, see Figure 3.3.

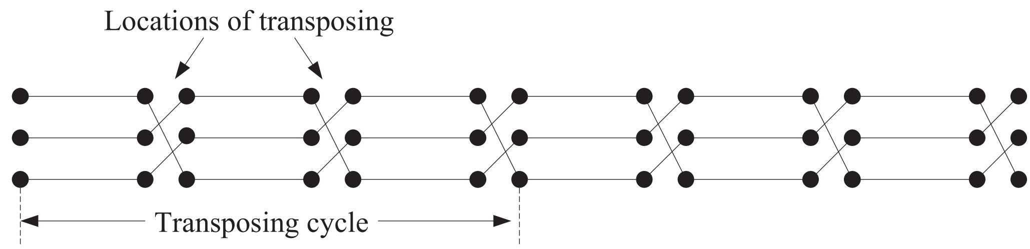


Figure 3.3. Transposing of three-phase overhead line.

This implies that each one of the conductors, under a transposing cycle, has had all three possible locations in the transmission line. Each location is held under equal distance which implies that all conductors in average have the same distance to ground and to the other conductors. This gives that the mutual inductance between the three phases are equalized so that the inductance per phase is equal among the three phases.

In many cases, the line is constructed as a multiple conductor, i.e. more than one conductor is used for each phase, see Figure 3.4. Multiple conductors implies both lower reactance of

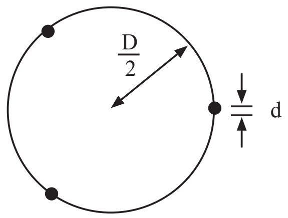


Figure 3.4. Cross-section of a multiple conductor with three conductors per phase.

the line and reduced corona effect (glow discharge). The radius $d/2$ in equation (3.2) must in these cases be replaced with the equivalent radius

$$(d/2)_{eq} = \sqrt[n]{n(D/2)^{n-1} \cdot (d/2)} \tag{3.3}$$

where

n = number of conductors per phase

$D/2$ = radius in the circle formed by the conductors

By using the inductance, the reactance of a line can be calculated as

$$x = \omega\, l = 2\pi f\, l \quad \Omega/\text{km,phase} \tag{3.4}$$

and is only dependent on the geometrical design of the line if the frequency is kept constant. The relationship between the geometrical mean distance a and the conductor diameter d in equation (3.2) varies within quite small limits for different lines. This due to the large distance between the phases and the larger conductor diameter for lines designed for higher system voltages. The term $\frac{1}{4n}$ has, compared with $\ln(\frac{a}{d/2})$, usually a minor influence on the line inductance.

At normal system frequency, the reactance of an overhead line can vary between 0.3 and 0.5 Ω/km,phase with a typical value of 0.4 Ω/km,phase. For cables, the reactance vary between 0.08 and 0.17 Ω/km,phase where the higher value is valid for cables with a small cross-section area. The reactance for cables is considerably lower than the reactance of overhead lines. The difference is caused by the difference in distance between the conductors. The conductors are more close to one another in cables which gives a lower reactance. See equation (3.2) which gives the inductance of overhead lines.

Example 3.1 *Determine the reactance of a 130 kV overhead line where the conductors are located in a plane and the distance between two closely located conductors is 4 m. The conductor diameter is 20 mm. Repeat the calculations for a line with two conductors per phase, located 30 cm from one another.*

Solution

$a_{12} = a_{23} = 4,\ a_{13} = 8$
$d/2 = 0.01$ m
$a = \sqrt[3]{4 \cdot 4 \cdot 8} = 5.04$
$x = 2\pi \cdot 50 \cdot 2 \cdot 10^{-4} \left(\ln \frac{5.04}{0.01} + \frac{1}{4}\right) = 0.0628 \left(\ln(504) + 0.25\right) = 0.41\ \Omega$/km,phase
Multiple conductor (duplex)
$(d/2)_{eq} = \sqrt[2]{2(0.3/2)0.01} = 0.055$
$x = 0.0628 \left(\ln \frac{5.04}{0.055} + \frac{1}{8}\right) = 0.29\ \Omega$/km,phase
The reactance is in this case reduced by 28 %.

3.1.4 Shunt capacitance

For a three-phase transposed overhead line, the capacitance to ground per phase can be calculated as

$$c = \frac{10^{-6}}{18 \ln \left(\frac{2H}{A} \cdot \frac{a}{(d/2)_{eq}} \right)} \text{ F/km,phase} \tag{3.5}$$

where

$H = \sqrt[3]{H_1 H_2 H_3}$ = geometrical mean height for the conductors according to Figure 3.2.

$A = \sqrt[3]{A_1 A_2 A_3}$ = geometrical mean distance between the conductors and their image conductors according to Figure 3.2.

As indicated in equation (3.5), the ground has some influence on the capacitance of the line. The capacitance is determined by the electrical field which is dependent on the characteristics of the ground. The ground will form an equipotential surface which has an influence on the electric field.

The degree of influence the ground has on the capacitance is determined by the factor $2H/A$ in equation (3.5). This factor has usually a value near 1.

Assume that a line mounted on relatively high poles ($\Rightarrow A \approx 2H$) is considered and that the term $\frac{1}{4n}$ can be neglected in equation (3.2). By multiplying the expressions for inductance and capacitance, the following is obtained

$$l \cdot c = 2 \cdot 10^{-4} \left(\ln \frac{a}{(d/2)_{eq}} \right) \cdot \frac{10^{-6}}{18 \ln \left(\frac{a}{(d/2)_{eq}} \right)} = \frac{1}{(3 \cdot 10^5)^2} \left(\frac{\text{km}}{\text{s}} \right)^{-2} = \frac{1}{v^2} \tag{3.6}$$

where v = speed of light in vacuum in km/s. Equation (3.6) can be interpreted as the inductance and capacitance are the inverse of one another for a line. Equation (3.6) is a good approximation for an overhead line. The shunt susceptance of a line is

$$b_c = 2\pi f \cdot c \qquad \text{S/km,phase} \tag{3.7}$$

A typical value of the shunt susceptance of a line is $3 \cdot 10^{-6}$ S/km,phase. Cables have considerable higher values between $3 \cdot 10^{-5} - 3 \cdot 10^{-4}$ S/km,phase.

Example 3.2 *Assume that a line has a shunt susceptance of* $3 \cdot 10^{-6}$ *S/km,phase. Use equation* (3.6) *to estimate the reactance of the line.*

Solution

$$x = \omega l \approx \frac{\omega}{cv^2} = \frac{\omega^2}{bv^2} = \frac{(100\pi)^2}{3 \cdot 10^{-6}(3 \cdot 10^5)^2} = 0.366 \quad \Omega/\text{km}$$

which is near the standard value of 0.4 Ω/km for the reactance of an overhead line.

3.2 Model of a line

Both overhead lines and cables have their electrical quantities r, x, g and b distributed along the whole length. Figure 3.1 shows an approximation of the distribution of the quantities. Generally, the accuracy of the calculation result will increase with the number of distributed quantities.

At a first glance, it seems possible to form a line model where the total resistance/inductance is calculated as the product between the resistance/inductance per length unit and the length of the line. This approximation is though only valid for short lines and lines of medium length. For long lines, the distribution of the quantities r, l, c and g must be taken into account. Such analysis can be carried out with help of differential calculus.

There are no absolute limits between short, medium and long lines. Usually, lines shorter than 100 km are considered as short, between 100 km and 300 km as medium long and lines longer than 300 km are classified as long. For cables, having considerable higher values of the shunt capacitance, the distance 100 km should be considered as medium long. In the following, models for short and medium long lines are given.

3.2.1 Short lines

In short line models, the shunt parameters are neglected, i.e. conductance and susceptance. This because the current flowing through these components is less than one percent of the rated current of the line. The short line model is given in Figure 3.5. This single-phase model of a three-phase system is valid under the assumption that the system is operating under symmetrical conditions.

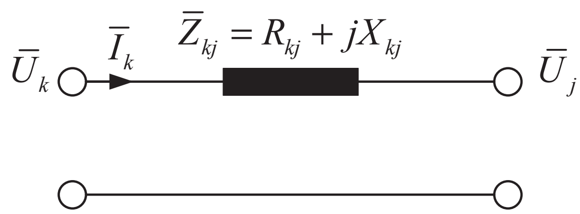


Figure 3.5. Short line model of a line.

The impedance of the line between bus k and bus j can be calculated as

$$\overline{Z}_{kj} = R_{kj} + jX_{kj} = (r_{kj} + jx_{kj})\,\mathcal{L} \quad \Omega/\text{phase} \tag{3.8}$$

where $\mathcal{L}$ is the length of the line in km.

3.2.2 Medium long lines

For lines having a length between 100 and 300 km, the shunt capacitance cannot be neglected. The model shown in Figure 3.5 has to be extended with the shunt susceptance, which results in a model called the π-equivalent shown in Figure 3.6. The impedance is calculated

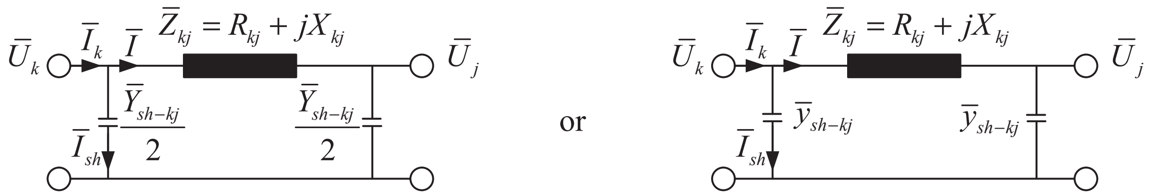


Figure 3.6. Medium long model of a line.

according to equation (3.8) and the admittance to ground per phase is obtained by

$$\frac{\overline{Y}_{sh-kj}}{2} = j\frac{b_c\,\mathcal{L}}{2} = \overline{y}_{sh-kj} = jb_{sh-kj} \quad \text{S} \tag{3.9}$$

i.e. the total shunt capacitance of the line is divided into two equal parts, one at each end of the line. The π-equivalent is a very common and useful model in power system analysis.

3.3 Single-phase transformer

The principle diagram of a two winding transformer is shown in Figure 3.7. The fundamental principles of a transformer are given in the figure. In a real transformer, the demand of a strong magnetic coupling between the primary and secondary sides must be taken into account in the design.

Assume that the magnetic flux can be divided into three components. There is a core flux Φ_m passing through both the primary and the secondary windings. There are also leakage fluxes, Φ_{l1} passing only the primary winding and Φ_{l2} which passes only the secondary winding. The resistance of the primary winding is r_1 and for the secondary winding r_2. According to the law of induction, the following relationships can be given for the voltages at the transformer terminals :

$$\begin{aligned} u_1 &= r_1 i_1 + N_1 \frac{d(\Phi_{l1} + \Phi_m)}{dt} \\ u_2 &= r_2 i_2' + N_2 \frac{d(\Phi_{l2} + \Phi_m)}{dt} \end{aligned} \tag{3.10}$$

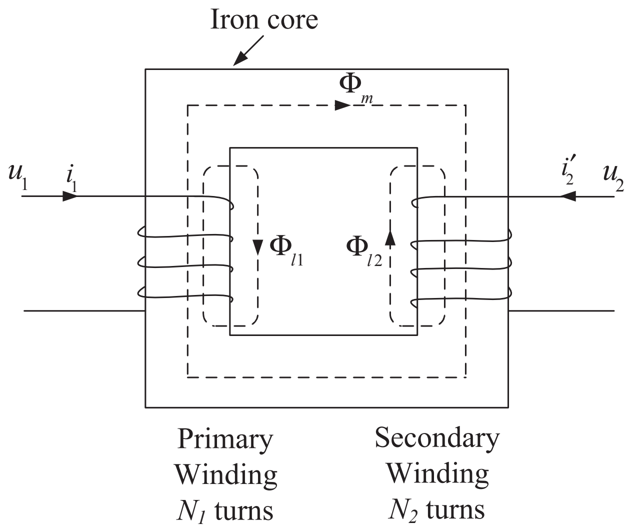


Figure 3.7. Principle design of a two winding transformer.

Assuming linear conditions, the following is valid

$$\begin{aligned} N_1\Phi_{l1} &= L_{l1}i_1 \\ N_2\Phi_{l2} &= L_{l2}i_2' \end{aligned} \tag{3.11}$$

where

$L_{l1} =$ inductance of the primary winding

$L_{l2} =$ inductance of the secondary winding

Equation (3.10) can be rewritten as

$$\begin{aligned} u_1 &= r_1 i_1 + L_{l1}\frac{di_1}{dt} + N_1\frac{d\Phi_m}{dt} \\ u_2 &= r_2 i_2' + L_{l2}\frac{di_2'}{dt} + N_2\frac{d\Phi_m}{dt} \end{aligned} \tag{3.12}$$

With the reluctance R of the iron core and the definitions of the directions of the currents according to Figure 3.7, the magnetomotive forces $N_1 i_1$ and $N_2 i_2'$ can be added as

$$N_1 i_1 + N_2 i_2' = R\Phi_m \tag{3.13}$$

Assume that $i_2' = 0$, i.e. the secondary side of the transformer is not connected. The current now flowing in the primary winding is called the *magnetizing current* and the magnitude can be calculated using equation (3.13) as

$$i_m = \frac{R\Phi_m}{N_1} \tag{3.14}$$

If equation (3.14) is inserted into equation (3.13), the result is

$$i_1 = i_m - \frac{N_2}{N_1} i_2' = i_m + \frac{N_2}{N_1} i_2 \tag{3.15}$$

where

$$i_2 = -i_2' \tag{3.16}$$

Assuming linear conditions, the induced voltage drop $N_1 \frac{d\Phi_m}{dt}$ in equation (3.12) can be expressed by using an inductor as

$$N_1 \frac{d\Phi_m}{dt} = L_m \frac{di_m}{dt} \tag{3.17}$$

i.e. $L_m = N_1^2/R$. By using equations (3.12), (3.15) and (3.17), the equivalent diagram of a single-phase transformer can be drawn, see Figure 3.8.

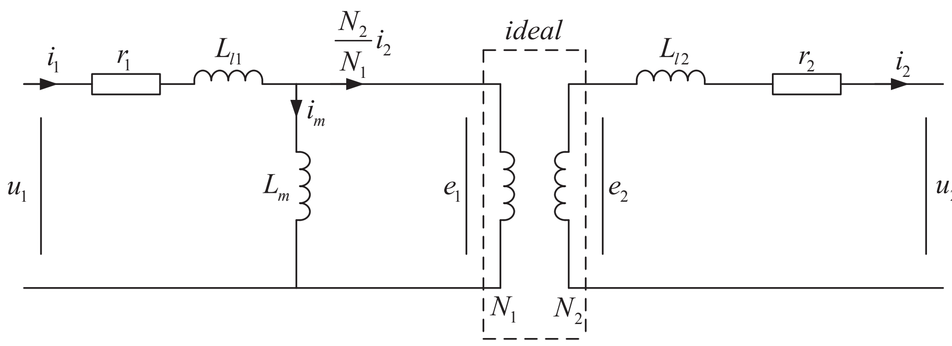


Figure 3.8. Equivalent diagram of a single-phase transformer.

In Figure 3.8, one part of the *ideal* transformer is shown, which is a lossless transformer without leakage fluxes and magnetizing currents.

The equivalent diagram in Figure 3.8 has the advantage that the different parts represents different parts of the real transformer. For example, the inductance L_m represents the assumed linear relationship between the core flux Φ_m and the magnetomotive force of the iron core. Also the resistive copper losses in the transformer are represented by r_1 and r_2.

In power system analysis, where the transformer is modeled, a simplified model is often used where the magnetizing current is neglected.

3.4 Three-phase transformer

There are three fundamental ways of connecting single-phase transformers into one three-phase transformer. The three combinations are Y-Y-connected, Δ-Δ-connected and Y-Δ-connected (or Δ-Y-connected). In Figure 3.9, the different combinations are shown.

When the neutral (i.e. n or N) is grounded, the Y-connected part will be designated by Y0. The different consequences that these different connections imply, will be discussed in Chapter 8.

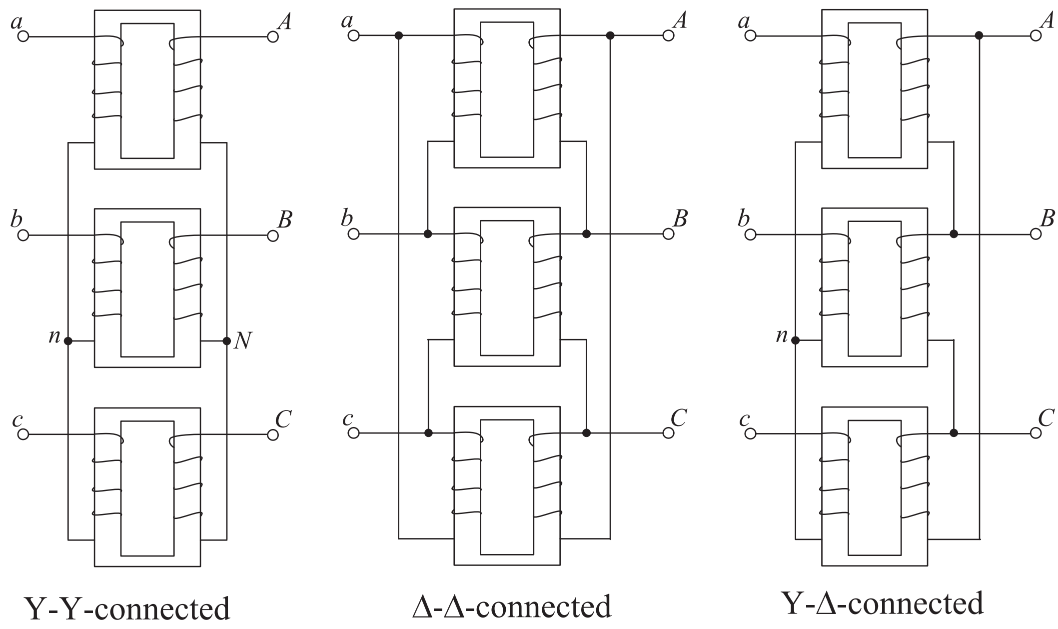


Figure 3.9. Standard connections for three-phase transformers.

3.4.1 Single-phase equivalent of three-phase transformers

Figure 3.10 shows the single-phase equivalent of a Y-Y-connected three-phase transformer. In the figure, $\overline{U}_{an}$ and $\overline{U}_{AN}$ are the line-to-neutral phasor voltages of the primary and secondary sides, respectively. However, $\overline{U}_{ab}$ and $\overline{U}_{AB}$ are the line-to-line phasor voltages of the primary and secondary sides, respectively. As shown in Figure 3.10 b), the ratio of line-to-neutral voltages is the same as the ratio of line-to-line voltages.

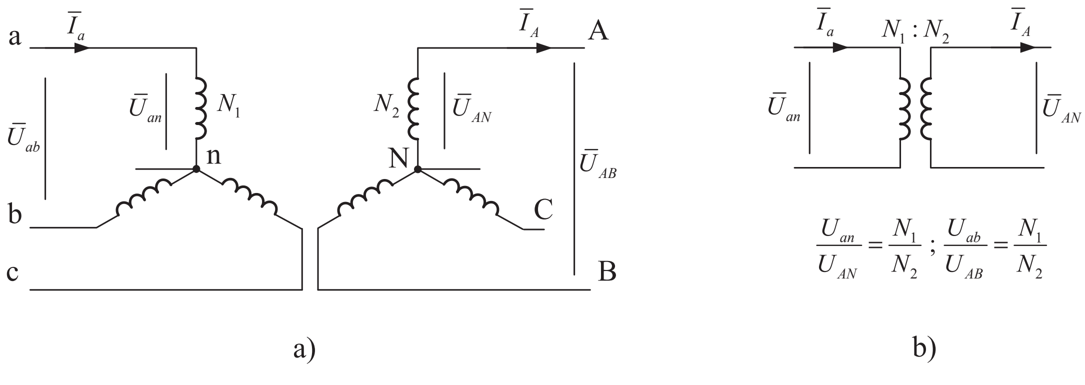


Figure 3.10. Single-phase equivalent of a three-phase Y-Y-connected transformer.

Figure 3.11 shows the single-phase equivalent of a Δ-Δ-connected three-phase transformer. For a Δ-Δ-connected transformer the ratio of line-to-neutral voltages is also the same as the ratio of line-to-line voltages. Furthermore, for Y-Y-connected and Δ-Δ-connected transformers $\overline{U}_{an}$ is in phase with $\overline{U}_{AN}$ (or $\overline{U}_{ab}$ is in phase with $\overline{U}_{AB}$).

It should be noted that Δ windings have no neutral, and for analysis of Δ-connected transformers it is more convenient to replace the Δ-connection with an equivalent Y-connection

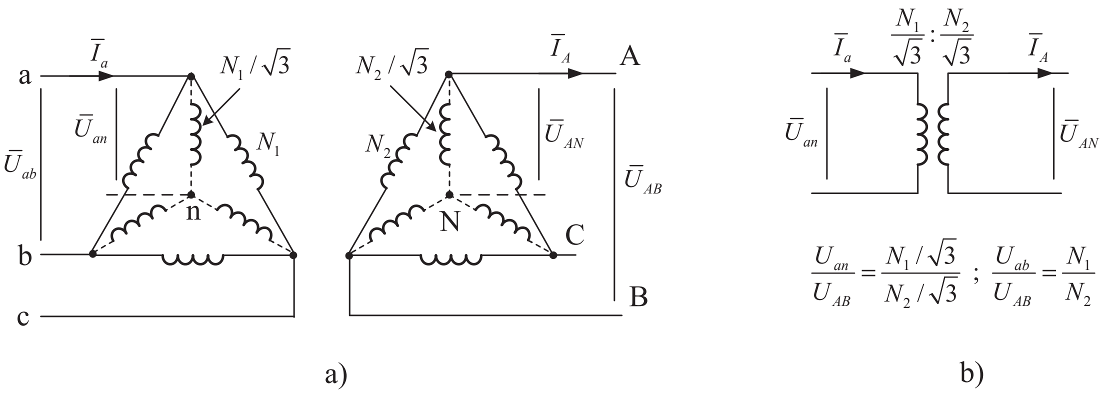


Figure 3.11. Single-phase equivalent of a three-phase Δ-Δ-connected transformer.

as shown with the dashed lines in the figure. Since for balanced operation, the neutrals of the equivalent Y-connections have the same potential the single-phase equivalent of both sides can be connected together by a neutral conductor. This is also valid for Y-Δ-connected (or Δ-Y-connected) three-phase transformer.

Figure 3.12 shows the single-phase equivalent of a Y-Δ-connected three-phase transformer.

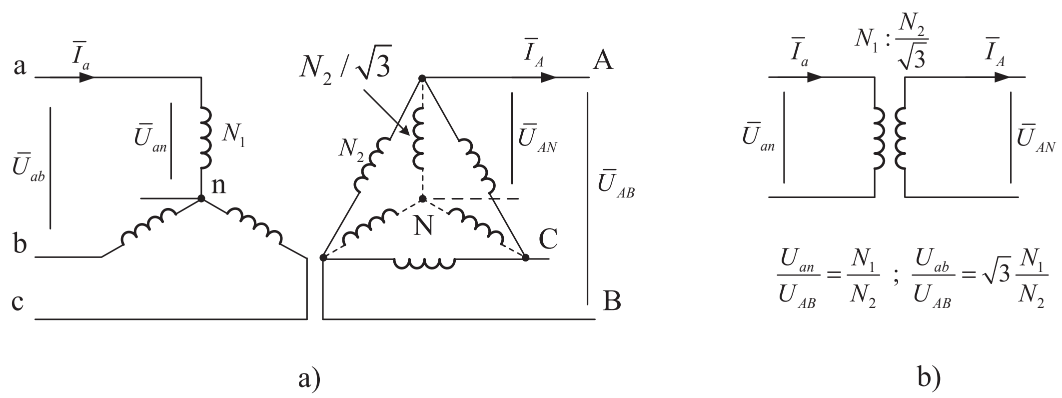


Figure 3.12. Single-phase equivalent of a three-phase Y-Δ-connected transformer.

It can be shown that $\overline{U}_{an} = \frac{N_1}{N_2}\,\overline{U}_{AB} = \sqrt{3}\,\frac{N_1}{N_2}\,\overline{U}_{AN}\,e^{j\,30^\circ}$, i.e. $\overline{U}_{an}$ leads $\overline{U}_{AN}$ by 30° (see also equation (2.15)).

In this compendium, this phase shift is not of concern. Furthermore, in this compendium the ratio of rated line-to-line voltages (rather than the turns ratio) will be used. Therefore, regardless of the transformer connection, the voltage and current can be transferred from the voltage level on one side to the voltage level on the other side by using the ratio of rated line-to-line voltages as multiplying factor. Also, the transformer losses and magnetizing currents (i.e. i_m in Figure 3.8) are neglected.

Figure 3.13 shows the single-line diagram of a lossless three-phase transformer which will be used in this compendium. In the figure, U_{1n} is the rated line-to-line voltage (given in kV) of the primary side and U_{2n} is the rated line-to-line voltage (given in kV) of the secondary side.

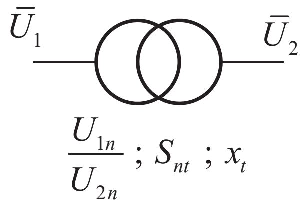


Figure 3.13. Single-line diagram of a three-phase transformer.

U_{1n}/U_{2n} is the ratio of rated line-to-line voltages. S_{nt} is the transformer three-phase rating given in MVA, and x_t is the transformer leakage reactance, normally given as a percent based on the transformer rated (or nominal) values. Finally, $\overline{U}_1$ and $\overline{U}_2$ are the line-to-line phasor voltages of the transformer terminals.

Chapter 4

Important theorems in power system analysis

In many cases, the use of theorems can simplify the analysis of electrical circuits and systems. In the following sections, some important theorems will be discussed and proofs will be given.

4.1 Bus analysis, admittance matrices

Consider an electric network which consists of four buses as shown in Figure 4.1. Each bus is connected to the other buses via an admittance $\overline{y}_{kj}$ where the subscript indicates which buses the admittance is connected to. Assume that there are no mutual inductances between

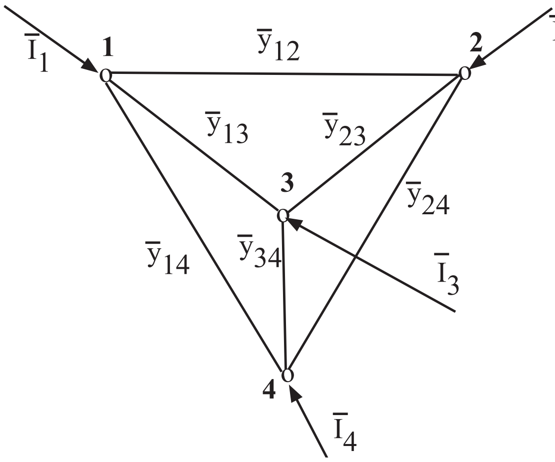


Figure 4.1. Four bus network.

the admittances and that the buses voltages are $\overline{U}_1$, $\overline{U}_2$, $\overline{U}_3$ and $\overline{U}_4$. The currents $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ and $\overline{I}_4$ are assumed to be injected into the buses from external current sources. Application of Kirchhoff's current law at bus 1 gives

$$\overline{I}_1 = \overline{y}_{12}(\overline{U}_1 - \overline{U}_2) + \overline{y}_{13}(\overline{U}_1 - \overline{U}_3) + \overline{y}_{14}(\overline{U}_1 - \overline{U}_4) \tag{4.1}$$

or

$$\begin{aligned} \overline{I}_1 &= (\overline{y}_{12} + \overline{y}_{13} + \overline{y}_{14})\overline{U}_1 - \overline{y}_{12}\overline{U}_2 - \overline{y}_{13}\overline{U}_3 - \overline{y}_{14}\overline{U}_4 = \\ &= \overline{Y}_{11}\overline{U}_1 + \overline{Y}_{12}\overline{U}_2 + \overline{Y}_{13}\overline{U}_3 + \overline{Y}_{14}\overline{U}_4 \end{aligned} \tag{4.2}$$

where

$$\overline{Y}_{11} = \overline{y}_{12} + \overline{y}_{13} + \overline{y}_{14}\ ,\ \overline{Y}_{12} = -\overline{y}_{12}\ ,\ \overline{Y}_{13} = -\overline{y}_{13} \text{ and } \overline{Y}_{14} = -\overline{y}_{14} \tag{4.3}$$

Corresponding equations can be formed for the other buses. These equations can be put together to a matrix equation as :

$$\mathbf{I} = \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \\ \overline{I}_4 \end{bmatrix} = \begin{bmatrix} \overline{Y}_{11}\overline{Y}_{12}\overline{Y}_{13}\overline{Y}_{14} \\ \overline{Y}_{21}\overline{Y}_{22}\overline{Y}_{23}\overline{Y}_{24} \\ \overline{Y}_{31}\overline{Y}_{32}\overline{Y}_{33}\overline{Y}_{34} \\ \overline{Y}_{41}\overline{Y}_{42}\overline{Y}_{43}\overline{Y}_{44} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \\ \overline{U}_4 \end{bmatrix} = \mathbf{YU} \tag{4.4}$$

This matrix is termed as the bus admittance matrix or Y-bus matrix which has the following properties :

- It can be uniquely determined from a given admittance network.
- The diagonal element $\overline{Y}_{kk}$ is the sum of all admittances connected to bus k.
- The non-diagonal element $\overline{Y}_{kj}$ is defined by $\overline{Y}_{kj} = -\overline{y}_{kj} = -\frac{1}{\overline{Z}_{kj}}$ where $\overline{y}_{kj}$ is the admittance between bus k and bus j.
- This gives that the matrix is symmetric, i.e. $\overline{Y}_{kj} = \overline{Y}_{jk}$ (one exception is when the network includes phase shifting transformers).
- It is singular since $\overline{I}_1 + \overline{I}_2 + \overline{I}_3 + \overline{I}_4 = 0$

If the potential in one bus is assumed to be zero, the corresponding row and column in the admittance matrix can be removed which results in a non-singular matrix. Bus analysis using the Y-bus matrix is the method most often used when studying larger, meshed networks in a systematic manner.

Example 4.1 *Re-do Example 2.5 by using the Y-bus matrix of the network in order to calculate the power given by the radiators.*

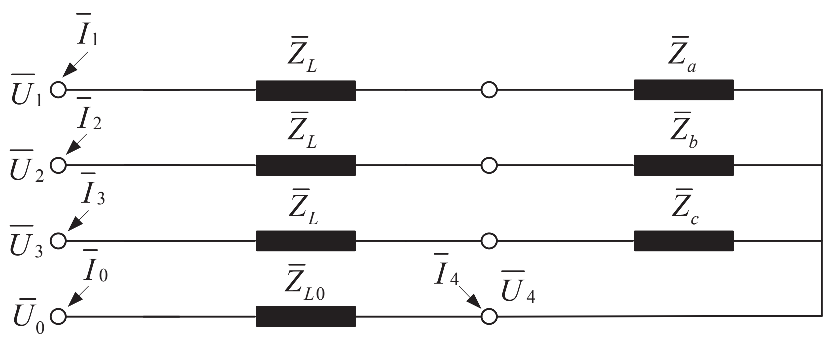


Figure 4.2. Network diagram used in the example.

Solution

According to the task and to the calculations performed in Example 2.5, the following is valid; $\overline{Z}_L = 2.3 + j0.16\ \Omega, \overline{Z}_{L0} = 2.3 + j0.03\ \Omega, \overline{Z}_a = 47.9 + j4.81\ \Omega, \overline{Z}_b = 15.97 + j1.60\ \Omega, \overline{Z}_c =$

$23.96 + j2.40\ \Omega$. Start with forming the Y-bus matrix. $\overline{I}_0$ and $\overline{U}_0$ are neglected since the system otherwise will be singular.

$$\mathbf{I} = \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \\ \overline{I}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\overline{Z}_L+\overline{Z}_a} & 0 & 0 & -\frac{1}{\overline{Z}_L+\overline{Z}_a} \\ 0 & \frac{1}{\overline{Z}_L+\overline{Z}_b} & 0 & -\frac{1}{\overline{Z}_L+\overline{Z}_b} \\ 0 & 0 & \frac{1}{\overline{Z}_L+\overline{Z}_c} & -\frac{1}{\overline{Z}_L+\overline{Z}_c} \\ -\frac{1}{\overline{Z}_L+\overline{Z}_a} & -\frac{1}{\overline{Z}_L+\overline{Z}_b} & -\frac{1}{\overline{Z}_L+\overline{Z}_c} & \overline{Y}_{44} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \\ \overline{U}_4 \end{bmatrix} = \mathbf{YU} \tag{4.5}$$

where

$$\overline{Y}_{44} = \frac{1}{\overline{Z}_L + \overline{Z}_a} + \frac{1}{\overline{Z}_L + \overline{Z}_b} + \frac{1}{\overline{Z}_L + \overline{Z}_c} + \frac{1}{\overline{Z}_{L0}} \tag{4.6}$$

In the matrix equation above, $\overline{U}_1, \overline{U}_2, \overline{U}_3$ and $\overline{I}_4$ ($\overline{I}_4$=0) as well as all impedances, i.e. the Y-bus matrix, are known. If the given Y-bus matrix is inverted, the corresponding Z-bus matrix is obtained :

$$\mathbf{U} = \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \\ \overline{U}_4 \end{bmatrix} = \mathbf{ZI} = \mathbf{Y^{-1}I} = \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13}\overline{Z}_{14} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23}\overline{Z}_{24} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33}\overline{Z}_{34} \\ \overline{Z}_{41}\overline{Z}_{42}\overline{Z}_{43}\overline{Z}_{44} \end{bmatrix} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \\ \overline{I}_4 \end{bmatrix} \tag{4.7}$$

Since the elements in the Y-bus matrix are known, all the elements in the Z-bus matrix can be calculated. Since $\overline{I}_4$=0 the voltages $\overline{U}_1, \overline{U}_2$ and $\overline{U}_3$ can be expressed as a function of the currents $\overline{I}_1, \overline{I}_2$ and $\overline{I}_3$ by using only a part of the Z-bus matrix :

$$\begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix} = \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} \tag{4.8}$$

Since the voltages $\overline{U}_1, \overline{U}_2$ and $\overline{U}_3$ are known, the currents $\overline{I}_1, \overline{I}_2$ and $\overline{I}_3$ can be calculated as :

$$\begin{aligned} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} &= \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix} = \\ &= 10^{-3} \begin{bmatrix} 19.0 - j1.83 & -1.95 - j0.324 & -1.36 + j0.227 \\ -1.95 + j0.324 & 48.9 - j4.36 & -3.73 - j0.614 \\ -1.36 + j0.227 & -3.73 + j0.614 & 35.1 - j3.25 \end{bmatrix} \begin{bmatrix} 220\angle 0^\circ \\ 220\angle -120^\circ \\ 220\angle 120^\circ \end{bmatrix} = \\ &= \begin{bmatrix} 4.58\angle -4.39^\circ \\ 11.5\angle -123.6^\circ \\ 8.31\angle 111.3^\circ \end{bmatrix} \text{ A} \end{aligned} \tag{4.9}$$

By using these currents, the power given by the radiators can be calculated as :

$$\begin{aligned} \overline{S}_{za} &= \overline{Z}_a I_1^2 = 1004 + j101 \text{ VA} \\ \overline{S}_{zb} &= \overline{Z}_b I_2^2 = 2095 + j210 \text{ VA} \qquad \sum = 4754 + j477 \text{ VA} \\ \overline{S}_{zc} &= \overline{Z}_c I_3^2 = 1655 + j166 \text{ VA} \end{aligned} \tag{4.10}$$

i.e. the thermal power obtained is 4754 W.

4.2 Millman's theorem

Millman's theorem (the parallel generator-theorem) gives that if a number of admittances $\overline{Y}_{1k}, \overline{Y}_{2k}, \overline{Y}_{3k} \ldots \overline{Y}_{nk}$ are connected to a common bus k, and the voltages to a reference bus $\overline{U}_{10}, \overline{U}_{20}, \overline{U}_{30} \ldots \overline{U}_{n0}$ are known, the voltage between bus k and the reference bus, $\overline{U}_{k0}$ can be calculated as

$$\overline{U}_{k0} = \frac{\sum_{i=1}^{n} \overline{Y}_{ik}\overline{U}_{i0}}{\sum_{i=1}^{n} \overline{Y}_{ik}} \tag{4.11}$$

Assume a Y-connection of admittances as shown in Figure 4.3. The Y-bus matrix for this

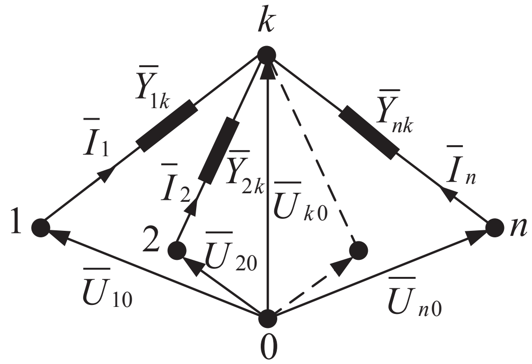


Figure 4.3. Y-connected admittances.

network can be formed as

$$\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \vdots \\ \overline{I}_n \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} \overline{Y}_{1k} & 0 & \ldots & 0 & -\overline{Y}_{1k} \\ 0 & \overline{Y}_{2k} & \ldots & 0 & -\overline{Y}_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \overline{Y}_{nk} & -\overline{Y}_{nk} \\ -\overline{Y}_{1k} & -\overline{Y}_{2k} & \ldots & -\overline{Y}_{nk} & (\overline{Y}_{1k} + \overline{Y}_{2k} + \ldots \overline{Y}_{nk}) \end{bmatrix} \begin{bmatrix} \overline{U}_{10} \\ \overline{U}_{20} \\ \vdots \\ \overline{U}_{n0} \\ \overline{U}_{k0} \end{bmatrix} \tag{4.12}$$

This equation can be written as

$$\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \vdots \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} \overline{U}_{10}\overline{Y}_{1k} - \overline{U}_{k0}\overline{Y}_{1k} \\ \overline{U}_{20}\overline{Y}_{2k} - \overline{U}_{k0}\overline{Y}_{2k} \\ \vdots \\ -\overline{U}_{10}\overline{Y}_{1k} - \overline{U}_{20}\overline{Y}_{2k} - \ldots + \sum_{i=1}^{n} \overline{Y}_{ik}\overline{U}_{k0} \end{bmatrix} \tag{4.13}$$

Since no current is injected at bus k ($\overline{I}_k = 0$), the last equation can be written as

$$\overline{I}_k = 0 = -\overline{U}_{10}\overline{Y}_{1k} - \overline{U}_{20}\overline{Y}_{2k} - \ldots + \sum_{i=1}^{n} \overline{Y}_{ik}\overline{U}_{k0} \tag{4.14}$$

This equation can be written as

$$\overline{U}_{k0} = \frac{\overline{U}_{10}\overline{Y}_{1k} + \overline{U}_{20}\overline{Y}_{2k} + \ldots + \overline{U}_{n0}\overline{Y}_{nk}}{\sum_{i=1}^{n} \overline{Y}_{ik}} \tag{4.15}$$

and by that, the proof of the Millman's theorem is completed.

Example 4.2 *Find the solution to Example 2.5 by using Millman's theorem, which will be the most efficient method to solve the problem so far.*

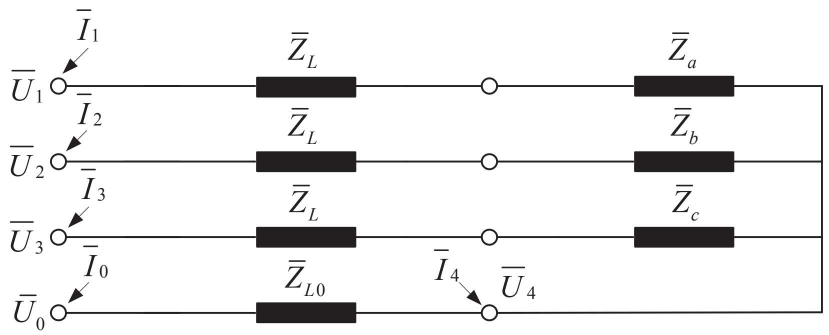


Figure 4.4. Diagram of the network used in the example.

Solution

According to the task and to the calculations performed in Example 2.5, the following is valid; $\overline{Z}_L = 2.3 + j0.16\ \Omega, \overline{Z}_{L0} = 2.3 + j0.03\ \Omega, \overline{Z}_a = 47.9 + j4.81\ \Omega, \overline{Z}_b = 15.97 + j1.60\ \Omega, \overline{Z}_c = 23.96 + j2.40\ \Omega$.

By using Millman's theorem (i.e. equation (4.15)), the voltage at bus 4 can be calculated by

$$\begin{aligned}\overline{U}_{40} &= \frac{\overline{U}_0\frac{1}{\overline{Z}_{L0}} + \overline{U}_1\frac{1}{\overline{Z}_a+\overline{Z}_L} + \overline{U}_2\frac{1}{\overline{Z}_b+\overline{Z}_L} + \overline{U}_3\frac{1}{\overline{Z}_c+\overline{Z}_L}}{\frac{1}{\overline{Z}_{L0}} + \frac{1}{\overline{Z}_a+\overline{Z}_L} + \frac{1}{\overline{Z}_b+\overline{Z}_L} + \frac{1}{\overline{Z}_c+\overline{Z}_L}} = \\ &= 12.08\angle - 155.1^\circ \text{ V}\end{aligned} \tag{4.16}$$

The currents through the impedances can be calculated as

$$\begin{aligned}\overline{I}_1 &= \frac{\overline{U}_1 - \overline{U}_4}{\overline{Z}_a + \overline{Z}_L} = 4.58\angle - 4.39^\circ \text{ A} \\ \overline{I}_2 &= \frac{\overline{U}_2 - \overline{U}_4}{\overline{Z}_b + \overline{Z}_L} = 11.5\angle - 123.6^\circ \text{ A} \\ \overline{I}_3 &= \frac{\overline{U}_3 - \overline{U}_4}{\overline{Z}_c + \overline{Z}_L} = 8.31\angle 111.3^\circ \text{ A}\end{aligned} \tag{4.17}$$

By using these currents, the power from the radiators can be calculated in the same way as earlier :

$$\begin{aligned} \overline{S}_{za} &= \overline{Z}_a I_1^2 = 1004 + j101 \text{ VA} \\ \overline{S}_{zb} &= \overline{Z}_b I_2^2 = 2095 + j210 \text{ VA} \qquad \sum = 4754 + j477 \text{ VA} \\ \overline{S}_{zc} &= \overline{Z}_c I_3^2 = 1655 + j166 \text{ VA} \end{aligned} \tag{4.18}$$

i.e. the thermal power is 4754 W.

4.3 Superposition theorem

According to section 4.1, each admittance network can be described by a Y-bus matrix, i.e.

$$\mathbf{I} = \mathbf{YU} \tag{4.19}$$

where

$$\begin{aligned} \mathbf{I} &= \text{vector with currents injected into the buses} \\ \mathbf{U} &= \text{vector with the bus voltages} \end{aligned}$$

The superposition theorem can be applied to variables with a linear dependence, as shown in equation (4.19). This implies that the solution is obtained piecewise, e.g. for one generator at the time. The total solution is obtained by adding all the part solutions found :

$$\mathbf{I} = \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \vdots \\ \overline{I}_n \end{bmatrix} = \mathbf{Y} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \vdots \\ \overline{U}_n \end{bmatrix} = \mathbf{Y} \begin{bmatrix} \overline{U}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \mathbf{Y} \begin{bmatrix} 0 \\ \overline{U}_2 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \mathbf{Y} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \overline{U}_n \end{bmatrix} \tag{4.20}$$

It can be noted that the superposition theorem cannot be applied to calculations of the power flow since they cannot be considered as linear properties since they are the product between voltage and current.

Example 4.3 *Use the conditions given in Example 4.1 and assume that a fault at the feeding transformer gives a short circuit of phase 2. Phase 1 and 3 are operating as usual. Calculate the thermal power obtained in the house of Elektra.*

Solution
According to equation (4.9) in Example 4.1, the phase currents can be expressed as a function of the feeding voltages as

$$\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} = \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix} \tag{4.21}$$

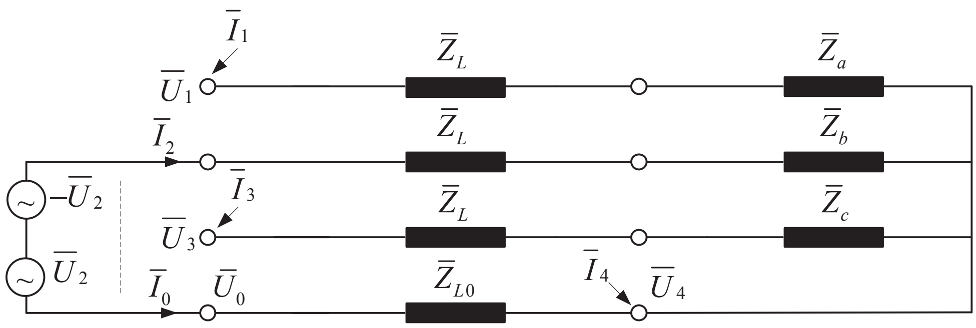


Figure 4.5. Diagram of the network used in the example.

A short circuit in phase 2 is equivalent with connecting an extra voltage source in reverse direction in series with the already existing voltage source. The phase currents in the changed system can be calculated as :

$$\begin{aligned}
\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} &= \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix} + \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\overline{U}_2 \\ 0 \end{bmatrix} = \\
&= \begin{bmatrix} 4.58\angle-4.39^\circ \\ 11.5\angle-123.6^\circ \\ 8.31\angle111.3^\circ \end{bmatrix} + \begin{bmatrix} \overline{Z}_{11}\overline{Z}_{12}\overline{Z}_{13} \\ \overline{Z}_{21}\overline{Z}_{22}\overline{Z}_{23} \\ \overline{Z}_{31}\overline{Z}_{32}\overline{Z}_{33} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -220\angle-120^\circ \\ 0 \end{bmatrix} = \\
&= \begin{bmatrix} 4.34\angle-9.09^\circ \\ 0.719\angle-100.9^\circ \\ 7.94\angle-116.5^\circ \end{bmatrix} \text{ A} \qquad (4.22)
\end{aligned}$$

$$\begin{aligned}
\overline{S}_{za} &= \overline{Z}_a I_1^2 = 904 + j91 \text{ VA} \\
\overline{S}_{zb} &= \overline{Z}_b I_2^2 = 8.27 + j0.830 \text{ VA} \qquad \sum = 2421 + j243 \text{ VA} \qquad (4.23) \\
\overline{S}_{zc} &= \overline{Z}_c I_3^2 = 1509 + j151 \text{ VA}
\end{aligned}$$

i.e. the thermal power is 2421 W

As shown in this example, the superposition theorem can, for instance, be used when studying changes in the system. But it should once again be pointed out that this is valid under the assumption that the loads (the radiators in this example) can be modeled as impedances.

4.4 Reciprocity theorem

Assume that a voltage source is connected to a terminal k in a linear reciprocal network and is giving rise to a current at terminal l. According to the reciprocity theorem, the voltage source will cause the same current at k if it is connected to l. The Y-bus matrix (and by that also the Z-bus matrix) are symmetrical matrices for a reciprocal electric network.

Assume that an electric network with n buses can be described by a symmetric Y-bus matrix, i.e.

$$\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \vdots \\ \overline{I}_n \end{bmatrix} = \mathbf{I} = \mathbf{YU} = \begin{bmatrix} \overline{Y}_{11} & \overline{Y}_{12} & \dots & \overline{Y}_{1n} \\ \overline{Y}_{21} & \overline{Y}_{22} & \dots & \overline{Y}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{Y}_{n1} & \overline{Y}_{n2} & \dots & \overline{Y}_{nn} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \vdots \\ \overline{U}_n \end{bmatrix} \tag{4.24}$$

Assume that $\overline{U}_k$ is the only non-zero voltage. The current at l can now be calculated as

$$\overline{I}_l = \overline{Y}_{lk}\overline{U}_k \tag{4.25}$$

Assume now that $\overline{U}_l$ is the only non-zero voltage. This means that the current at k is

$$\overline{I}_k = \overline{Y}_{kl}\overline{U}_l \tag{4.26}$$

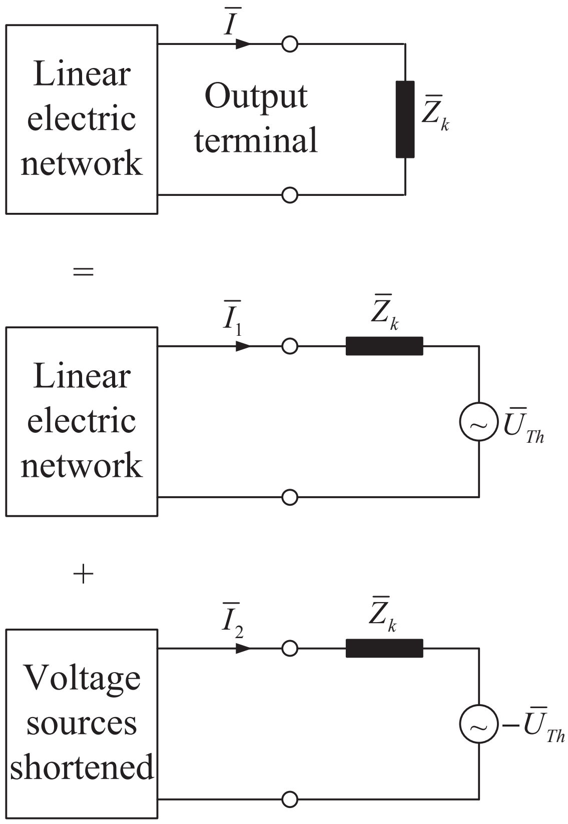
If $\overline{U}_k = \overline{U}_l$, the currents $\overline{I}_k$ and $\overline{I}_l$ will be equal since the Y-bus matrix is symmetric, i.e. $\overline{Y}_{kl} = \overline{Y}_{lk}$. By that, the proof of the reciprocity theorem is completed.

4.5 Thévenin-Helmholtz's theorem

This theorem is often called the Thévenin's theorem (after Léon Charles Thévenin, telegraph engineer and teacher, who published the theorem in 1883). But 30 years earlier, Hermann von Helmholtz published the same theorem in 1853, including a simple proof. The theorem can be described as follows:

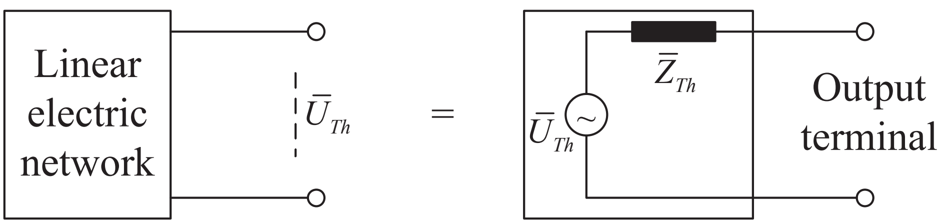
- Thévenin-Helmholtz's theorem states that from any output terminal in a linear electric network, no matter how complex, the entire linear electric network as seen from the output terminal can be modelled as an ideal voltage source $\overline{U}_{Th}$ (i.e. the voltage will be constant (or unchanged) regardless of how the voltage source is loaded) in series with an impedance $\overline{Z}_{Th}$. According to this theorem, when the output terminal is not loaded, its voltage is $\overline{U}_{Th}$, and the impedance $\overline{Z}_{Th}$ is the impedance as seen from the output terminal when all voltage sources in the network are short circuited and all current sources are disconnected.

Proof :
Assume that the voltage at an output terminal is $\overline{U}_{Th}$. Loading the output terminal with an impedance $\overline{Z}_k$, a current $\overline{I}$ will flow through the impedance. This connection is similar to have a network with a voltage source $\overline{U}_{Th}$ connecting to the output terminal in series with the impedance $\overline{Z}_k$, together with having a network with the voltage source $-\overline{U}_{Th}$ connecting to the output terminal and the other voltage sources in the network shortened. By using the superposition theorem, the current $\overline{I}$ can be calculated as the sum of $\overline{I}_1$ and $\overline{I}_2$. The current $\overline{I}_1 = 0$ since the voltage is equal on both sides of the impedance $\overline{Z}_k$. The current $\overline{I}_2$ can be calculated as
$\overline{I}_2 = -(-\overline{U}_{Th})/(\overline{Z}_k + \overline{Z}_{Th})$
since the network impedance seen from the output terminal is $\overline{Z}_{Th}$. The conclusion is that



$$\overline{I} = \overline{I}_1 + \overline{I}_2 = \frac{\overline{U}_{Th}}{\overline{Z}_k + \overline{Z}_{Th}} \tag{4.27}$$

which is the same as stated by Thévenin-Helmholtz's theorem, viz.



Chapter 5

Analysis of balanced three-phase systems

Consider the simple balanced three-phase system shown in Figure 5.1, where a symmetric three-phase Y0-connected generator supplies a symmetric Y0-connected impedance load. The neutral of the generator (i.e. point N) is grounded via the impedance $\overline{Z}_{NG}$. However, the neutral of the load (i.e. point n) is directly grounded. Since we are dealing with a balanced (or symmetrical) system, $\overline{I}_N = \overline{I}_a + \overline{I}_b + \overline{I}_c = \overline{I}_n = 0$, i.e. $\overline{U}_N = \overline{U}_n = 0$ and $\overline{Z}_{NG}$ has no impact on the system. Note that also in case of connecting point n directly to point N via the impedance $\overline{Z}_{NG}$, the neutrals n and N have the same potential, i.e. $\overline{U}_N = \overline{U}_n$, since in a balanced system $\overline{I}_a + \overline{I}_b + \overline{I}_c = 0$.

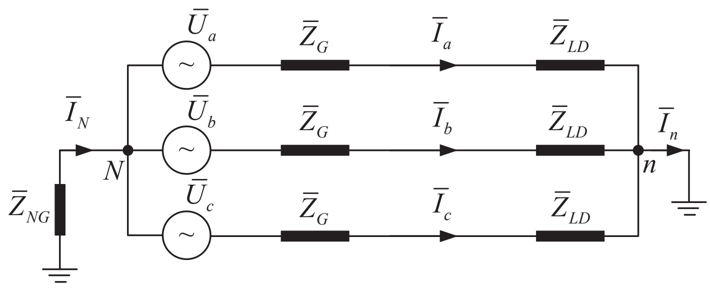


Figure 5.1. A simple three-phase system.

Therefore, the analysis of a balanced three-phase system can be carried out by studying only one single phase where the components can be connected together by a common neutral conductor as shown in Figure 5.1 a).

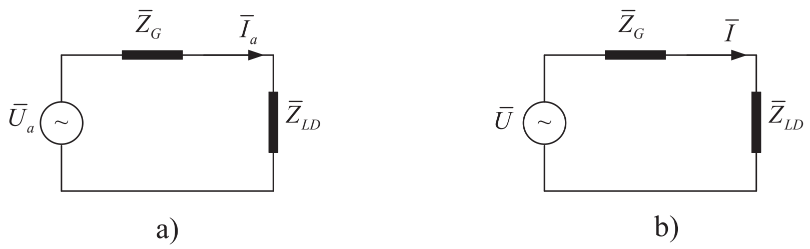


Figure 5.2. Single-phase equivalent of a symmetric three-phase system.

Based on Figure 5.1 a), the total three-phase supplied power is given by

$$\begin{aligned} \overline{I}_a &= I\, e^{j\gamma} = \frac{\overline{U}_a}{\overline{Z}_G + \overline{Z}_L} = \frac{U_{LN}\, e^{j\theta}}{\overline{Z}_G + \overline{Z}_L} \\ \overline{S}_{3\Phi} &= 3\,\overline{U}_a\,\overline{I}_a^* = 3\,U_{LN}\,I\,e^{j(\theta-\gamma)} = 3\,U_{LN}\,I\,e^{j\phi} \\ &= \sqrt{3}\,\sqrt{3}\,U_{LN}\,I\,e^{j\phi} = \sqrt{3}\,U_{LL}\,I\,e^{j\phi} \end{aligned} \tag{5.1}$$

For analysis of balanced three-phase systems, it is common to use the line-to line voltage magnitudes, i.e. the voltage $\overline{U}_a$ in Figure 5.1 a) is replaced by $\overline{U} = U\, e^{j\theta}$ (as shown in

Figure 5.1 b)) where, $U = U_{LL}$, however the phase angle of this voltage is the phase angle of the phase voltage. Furthermore, the other components in Figure 5.1 b) are per phase components. Based on Figure 5.1 b), we have then

$$\begin{aligned} \overline{U} &= \sqrt{3}\,\overline{I}\,(\overline{Z}_G + \overline{Z}_L) \\ \overline{S}_{3\Phi} &= \sqrt{3}\,\overline{U}\,\overline{I}^* = \sqrt{3}\,U\,I\,e^{j\phi} \end{aligned} \tag{5.2}$$

5.1 Single-line and impedance diagrams

A single-line diagram of a balanced three-phase power system shows the main components as well as the connections between them. A component is only given in the diagram if it is of interest for the analysis. Figure 5.3 shows the single-line diagram of a simple balanced three-phase power system. The system consists of four buses (or nodes) numbering from one to four, two generators G1 and G2, two transformers T1 and T2, two loads LD1 and LD2, and a transmission line between bus2 and bus3.

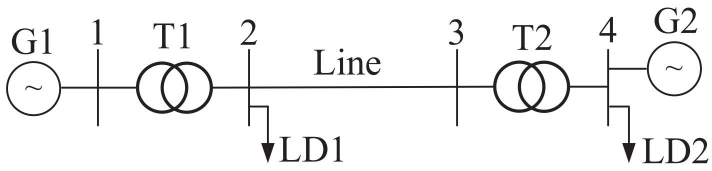


Figure 5.3. Single-line diagram of a small power system.

Here-onward, if not otherwise explicitly stated, the following is valid in this compendium:

- all system quantities (power, voltage, current, impedances and admittances) are given in the complex form,
- power is given as three-phase power in MVA, MW and/or MVAr,
- for the phasor voltage $\overline{U} = U\,\angle\theta$, the magnitude U is a line-to-line voltage given in kV, however the phase angle θ is the phase angle of a line-to-neutral voltage,
- currents (given in kA), impedances (given in Ω) and admittances (given in S) are per phase quantities.

Consider again the system shown in Figure 5.3. A typical system data can be given as follows:

- **Generator G1 :** S_{ng}=30 MVA, U_{ng}=10 kV, x_g=10%
- **Generator G2 :** S_{ng}=15 MVA, U_{ng}=6 kV, x_g=8%
- **Transformer T1 :** S_{nt}=15 MVA, $\dfrac{U_{1n}}{U_{2n}} = \dfrac{10\,kV}{30\,kV}$, x_t=10%

- **Transformer T2 :** S_{nt}=15 MVA, $\dfrac{U_{1n}}{U_{2n}} = \dfrac{30\,kV}{6\,kV}$, x_t=10%
- **Line :** $r = 0.17\ \Omega$/km, $x = 0.3\ \Omega$/km, $b_c = 3.2 \times 10^{-6}\ S$/km and $\mathcal{L} = 10$ km
- **Load LD1 :** impedance load, $P_{LD} = 15$ MW, $U_n = 30$ kV, $\cos\phi = 0.9$ inductive
- **Load LD2 :** impedance load, $P_{LD} = 40$ MW, $U_n = 6$ kV, $\cos\phi = 0.8$ inductive

Comments:

S_{ng} is the generator three-phase rating, U_{ng} is the generator rated (or nominal) line-to-line voltage and x_g is the generator reactance given as a percent based on the generator rated values. The actual value of the generator reactance can be determined by

$$X_g = \frac{x_g}{100}\frac{U_{ng}^2}{S_{ng}} \quad \Omega \quad \text{and} \quad \overline{Z}_g = j\,X_g$$

In a similar way the actual value of the transformer leakage reactance can be determined, however, depending on which side of the transformer it will be calculated. Having the reactance on the primary side, then it is determined by

$$X_{tp} = \frac{x_t}{100}\frac{U_{1n}^2}{S_{nt}} \quad \Omega \quad \text{and} \quad \overline{Z}_{tp} = j\,X_{tp}$$

Having the reactance on the secondary side, then it is determined by

$$X_{ts} = \frac{x_t}{100}\frac{U_{2n}^2}{S_{nt}} \quad \Omega \quad \text{and} \quad \overline{Z}_{ts} = j\,X_{ts}$$

For the line, using the model shown in Figure 3.6, we have

$$\overline{Z}_{12} = \mathcal{L}\,(r + jx) \quad \Omega \quad \text{and} \quad \overline{Y}_{sh-12} = jb_c\,\mathcal{L} \quad \text{S}$$

For the load, P is the consumed three-phase active power with the power factor $\cos\phi$ at the nominal (or rated) voltage U_n. Thus, the impedance load can be determined by

$$\overline{Z}_{LD} = \frac{U_n^2}{\overline{S}_{LD}^*} = \frac{U_n^2}{S_{LD}(\cos\phi - j\sin\phi)} = \frac{U_n^2}{S_{LD}}(\cos\phi + j\sin\phi) \quad \text{where} \quad S_{LD} = \frac{P_{LD}}{\cos\phi}$$

Figure 5.4 shows the single-phase impedance diagram corresponding to the single-line diagram shown in Figure 5.3.

The simple system shown in Figure 5.4 has three different voltage levels (6, 10 and 30 kV). The analysis of the system can be carried out by transferring all impedances to a single voltage level. This method gives often quite extensive calculations, especially dealing with large systems with several different voltage levels. To overcome this difficulty, the so called *per-unit* system was developed, and it will be presented in the next section.

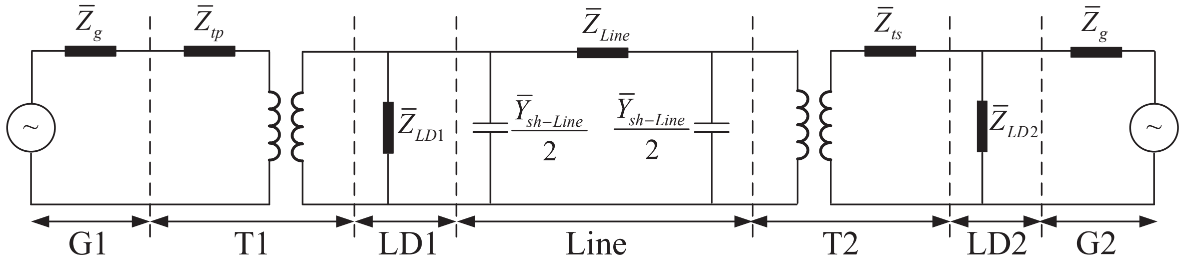


Figure 5.4. Impedance network of a small power system.

5.2 The per-unit (pu) system

A common method to express voltages, currents, powers and impedances in an electric network is in per-unit (or percent) of a certain base or reference value. The per-unit value of a certain quantity is defined as

$$\text{Per-unit value} = \frac{\text{true value}}{\text{base value of the quantity}} \tag{5.3}$$

The per-unit method is very suitable for power systems with several voltage levels and transformers. In a three-phase system, the per-unit value can be calculated using the corresponding base quantity. By using the base voltage

$$U_{base} = \text{base voltage, kV} \quad \text{(line-to-line voltage)} \tag{5.4}$$

and a base power,

$$S_{base} = \text{three-phase base power, MVA} \tag{5.5}$$

the base current

$$I_{base} = \frac{S_{base}}{\sqrt{3}\,U_{base}} = \text{base current/phase, kA} \tag{5.6}$$

as well as a base impedance

$$Z_{base} = \frac{U_{base}^2}{S_{base}} = \frac{U_{base}}{\sqrt{3}\,I_{base}} = \text{base impedance, } \Omega \tag{5.7}$$

can be calculated. In expressions given above, the units kV and MVA have been assumed, which imply units in kA and Ω. Of course, different combinations of units can be used, e.g. V, VA, A, Ω or kV, kVA, A, kΩ.

There are several reasons for using a per-unit system:

- The percentage voltage drop is directly given in the per-unit voltage.
- It is possible to analyze power systems having different voltage levels in a more efficient way.
- When having different voltage levels, the relative importance of different impedances is directly given by the per-unit value.

- When having large systems, numerical values of the same magnitude are obtained which increase the numerical accuracy of the analysis.
- Use of the constant $\sqrt{3}$ is reduced in three-phase calculations.

5.2.1 Per-unit representation of transformers

Figure 5.5 shows the single-phase impedance diagram of a symmetrical three-phase transformer. In Figure 5.5 a), the transformer leakage impedance is given on the primary side, and in Figure 5.5 b), the transformer leakage impedance is given on the secondary side. Furthermore, α is the ratio of rated line-to-line voltages. Thus, based on transformer properties we have

$$\frac{U_{1n}}{U_{2n}} = \frac{1}{\alpha} \quad \text{and} \quad \frac{\overline{I}_1}{\overline{I}_2} = \alpha \tag{5.8}$$

Let the base power be S_{base}. Note that S_{base} is a global base value, i.e. it is the same in all different voltages levels. Let also U_{1base} and U_{2base} be the base voltages on the primary side and secondary side, respectively. The base voltages have been chosen such that they have the same ratio as the ratio of the transformer, i.e.

$$\frac{U_{1base}}{U_{2base}} = \frac{1}{\alpha} \tag{5.9}$$

Furthermore, since $S_{base} = \sqrt{3}\,U_{1base}\,I_{1base} = \sqrt{3}\,U_{2base}\,I_{2base}$, by virtue of equation (5.9) we find that

$$\frac{I_{1base}}{I_{2base}} = \alpha \tag{5.10}$$

where, I_{1base} and I_{2base} are the base currents on the primary side and secondary side, respectively.

The base impedances on both sides are given by

$$Z_{1base} = \frac{U_{1base}^2}{S_{base}} = \frac{U_{1base}}{\sqrt{3}\,I_{1base}} \quad \text{and} \quad Z_{2base} = \frac{U_{2base}^2}{S_{base}} = \frac{U_{2base}}{\sqrt{3}\,I_{2base}} \tag{5.11}$$

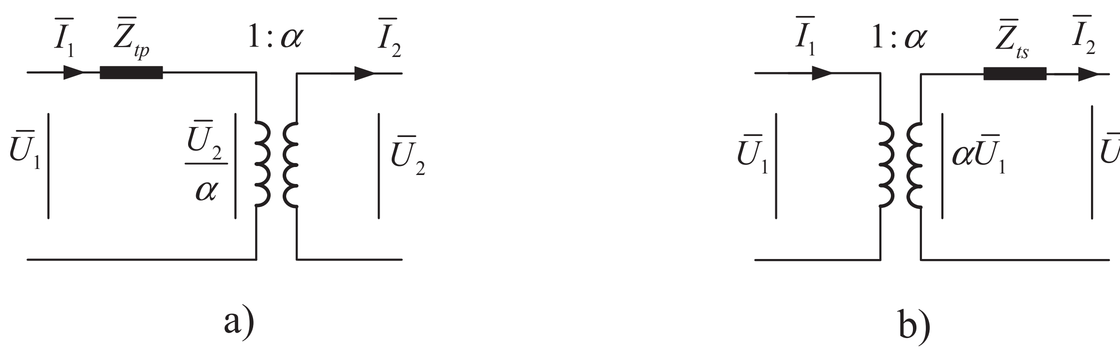


Figure 5.5. single-phase impedance diagram of a symmetrical three-phase transformer.

Now consider the circuit shown in Figure 5.5 a). The voltage equation is given by

$$\overline{U}_1 = \sqrt{3}\,\overline{I}_1\,\overline{Z}_{tp} + \frac{\overline{U}_2}{\alpha} \tag{5.12}$$

In per-unit (pu), we have

$$\frac{\overline{U}_1}{U_{1base}} = \frac{\sqrt{3}\,\overline{I}_1\,\overline{Z}_{tp}}{\sqrt{3}\,I_{1base}\,Z_{1base}} + \frac{\overline{U}_2}{\alpha\,U_{1base}} = \frac{\overline{I}_1}{I_{1base}}\,\frac{\overline{Z}_{tp}}{Z_{1base}} + \frac{\overline{U}_2}{U_{2base}} \quad \Rightarrow \quad \overline{U}_{1pu} = \overline{I}_{1pu}\,\overline{Z}_{tppu} + \overline{U}_{2pu} \tag{5.13}$$

Next, consider the circuit shown in Figure 5.5 b). The voltage equation is given by

$$\alpha\,\overline{U}_1 = \sqrt{3}\,\overline{I}_2\,\overline{Z}_{ts} + \overline{U}_2 \tag{5.14}$$

In per-unit (pu), we have

$$\frac{\alpha\,\overline{U}_1}{U_{2base}} = \frac{\alpha\,\overline{U}_1}{\alpha\,U_{1base}} = \frac{\sqrt{3}\,\overline{I}_2\,\overline{Z}_{ts}}{\sqrt{3}\,I_{2base}\,Z_{2base}} + \frac{\overline{U}_2}{U_{2base}} = \frac{\overline{I}_2}{I_{2base}}\,\frac{\overline{Z}_{ts}}{Z_{2base}} + \frac{\overline{U}_2}{U_{2base}} \quad \Rightarrow \quad \overline{U}_{1pu} = \overline{I}_{2pu}\,\overline{Z}_{tspu} + \overline{U}_{2pu} \tag{5.15}$$

By virtue of equations (5.13) and (5.15), we find that

$$\overline{I}_{1pu}\,\overline{Z}_{tppu} = \overline{I}_{2pu}\,\overline{Z}_{tspu}$$

Furthermore, based on equations (5.8) and (5.10) it can be shown that $\overline{I}_{1pu} = \overline{I}_{2pu}$ (show that). Thus,

$$\overline{Z}_{tppu} = \overline{Z}_{tspu} \tag{5.16}$$

Equation (5.16) implies that the per-unit impedance diagram of a transformer is the same regardless of whether the actual impedance is determined on the primary side or on the secondary side. Based on this property, the single-phase impedance diagram of a three-phase transformer in per-unit can be drawn as shown in Figure 5.6, where $\overline{Z}_{tpu} = \overline{Z}_{tppu} = \overline{Z}_{tspu}$.

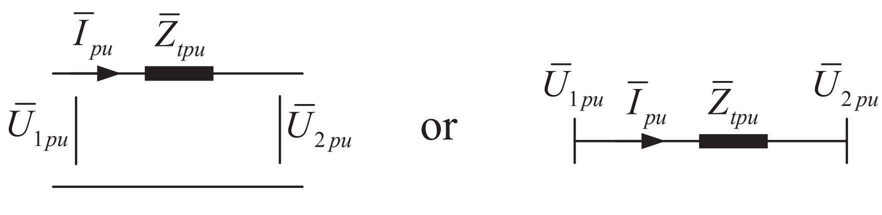


Figure 5.6. Per-unit impedance diagram of a transformer.

Example 5.1 *Assume that a 15 MVA transformer has a voltage ratio of 6 kV/30 kV and a leakage reactance of 8 %. Calculate the pu-impedance when the base power of the system is 20 MVA and the base voltage on the 30 kV-side is 33 kV.*

Solution

Based on given data, $S_{nt} = 15$ MVA, $U_{1n}/U_{2n} = 6/30$, $x_t = 8\%$ and $U_{2base} = 33$ kV. We first calculate the transformer impedance in ohm on the 30 kV-side and after that, the per-unit value.

$$\begin{aligned}\overline{Z}_{30kv} &= \frac{\overline{Z}_{\%}}{100} Z_{tbase30} = \frac{\overline{Z}_{\%}}{100}\frac{U_{2n}^2}{S_{nt}} = \frac{j8 \cdot 30^2}{100 \cdot 15} = j4.8\ \Omega \\ \overline{Z}_{tpu} &= \frac{\frac{\overline{Z}_{\%}}{100} Z_{tbase30}}{Z_{2base}} = \frac{\overline{Z}_{30kV}}{Z_{2base}} = \frac{\overline{Z}_{30kV} \cdot S_{base}}{U_{2base}^2} = \frac{j4.8 \cdot 20}{33^2} = j0.088\ pu\end{aligned}$$

The given leakage reactance in percent can be considered as the per unit value of reactance based on the transformer ratings, i.e. Z_{tbase}. To convert this per unit value to the system per unit value, we may apply the following equation

$$\overline{Z}_{tpu-new} = \overline{Z}_{tpu-given} \frac{U_{base-given}^2}{S_{base-given}} \frac{S_{base-new}}{U_{base-new}^2}$$

In our case, $\overline{Z}_{tpu-given} = j\,8/100$, $U_{base-given} = U_{2n} = 30$, $S_{base-given} = S_{nt} = 15$, $S_{base-new} = 20$, and $U_{base-new} = 33$.

Thus,

$$\overline{Z}_{tpu-new} = \frac{j8}{100}\frac{30^2}{15}\frac{20}{33^2} = j0.088\ pu \tag{5.17}$$

The pu-value of the reactance can be also determined based on the base values on the primary side. From equation (5.9), we have

$$\begin{aligned}\frac{U_{1base}}{U_{2base}} &= \frac{1}{\alpha} = \frac{6}{30} \quad \Rightarrow \quad U_{1base} = \frac{6}{30} U_{2base} = \frac{6}{30} 33 \\ Z_{1base} &= \frac{U_{1base}^2}{S_{base}} = \left(\frac{1}{\alpha}\right)^2 \frac{U_{2base}^2}{S_{base}} = \left(\frac{1}{\alpha}\right)^2 Z_{2base}\end{aligned}$$

Thus,

$$\begin{aligned}\overline{Z}_{6kv} &= \frac{\overline{Z}_{\%}}{100} Z_{tbase6} = \frac{\overline{Z}_{\%}}{100}\frac{U_{1n}^2}{S_{nt}} = \frac{j8}{100}\frac{6^2}{15} \\ \overline{Z}_{tpu} &= \frac{\overline{Z}_{6kV}}{Z_{1base}} = \frac{j8}{100}\frac{6^2}{15}\left(\frac{30}{6}\right)^2 \frac{20}{33^2} = \frac{j8}{100}\frac{30^2}{15}\frac{20}{33^2} = j0.088\ pu\end{aligned}$$

5.2.2 Per-unit representation of transmission lines

Figure 5.7 shows the π-equivalent model of a line, where $\overline{y}_{sh-kj} = \overline{Y}_{sh-kj}/2$.

The voltage at bus k in kV is given by

$$\overline{U}_k = \sqrt{3}\,\overline{Z}_{kj}\,\overline{I} + \overline{U}_j, \quad \text{where} \quad \overline{I} = \overline{I}_k - \overline{I}_{sh} = \overline{I}_k - \overline{y}_{sh-kj}\frac{\overline{U}_k}{\sqrt{3}}$$

Let S_{base}, U_{base}, I_{base} and Z_{base} be the base values for the line. Note that the base admittance is given by $Y_{base} = 1/Z_{base}$. Then, the above equations i per unit are given by

$$\frac{\overline{U}_k}{U_{base}} = \frac{\sqrt{3}\,\overline{Z}_{kj}\,\overline{I}}{\sqrt{3}\,Z_{base}\,I_{base}} + \frac{\overline{U}_j}{U_{base}} \quad \Rightarrow \quad \overline{U}_{kpu} = \overline{Z}_{kjpu}\,\overline{I}_{pu} + \overline{U}_{jpu}$$

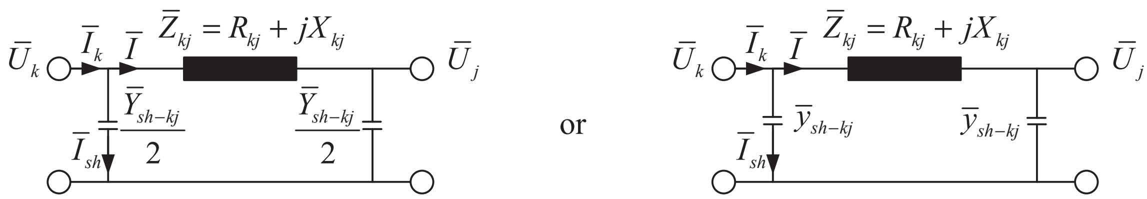


Figure 5.7. π-equivalent model of a line.

where

$$\overline{I}_{pu} = \frac{\overline{I}_k}{I_{base}} - \overline{y}_{sh-kj}\,\frac{\overline{U}_k}{\sqrt{3}}\,\frac{\sqrt{3}\,Z_{base}}{U_{base}} = \overline{I}_{kpu} - \overline{y}_{sh-kj}\,Z_{base}\,\frac{\overline{U}_k}{U_{base}} = \overline{I}_{kpu} - \overline{y}_{sh-kjpu}\,\overline{U}_{kpu}$$

Figure 5.8 shows the per-unit impedance diagram of a transmission line.

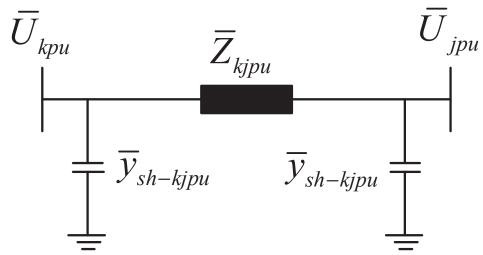


Figure 5.8. Per-unit impedance diagram of a transmission line.

5.2.3 System analysis in the per-unit system

To analyze a three-phase power system, it is more convenient and effective to convert the physical quantities into the per-unit system as follows:

1. Choose a suitable base power for the system. It should be in the same range as the rated power of the installed system equipments.
2. Choose a base voltage at one section (or voltage level) of the system. The system is divided into different sections (or voltage levels) by the transformers.
3. Calculate the base voltages in all sections of the system by using the transformer ratios.
4. Calculate all per-unit values of all system components that are connected.
5. Draw the per-unit impedance diagram of the system.
6. Perform the system analysis (in the per-unit system).
7. Convert the per-unit results back to the physical values.

Example 5.2 *Consider the power system shown in Figure 5.9, where a load is fed by a generator via a transmission line and two transformers. Based on the given system data below, calculate the load voltage as well as the active power of the load.*

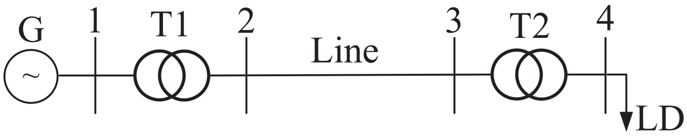


Figure 5.9. Single-line diagram of the system in Example 5.2.

System data:

Generator G : U_g=13.8 kV,
Transformer T1 : S_{nt}=10 MVA, $\dfrac{U_{1n}}{U_{2n}} = \dfrac{13.8\,kV}{69\,kV}$, X_{tp}=1.524 Ω (on 13.8 kV-side),
Transformer T2 : S_{nt}=5 MVA, $\dfrac{U_{1n}}{U_{2n}} = \dfrac{66\,kV}{13.2\,kV}$, x_t=8%,
Line : $x = 0.8\ \Omega$/km and $\mathcal{L} = 10$ km
Load LD : impedance load, $P_{LD} = 4$ MW, $U_n = 13.2$ kV, $\cos\phi = 0.8$ inductive.

Solution
1. Let the base power be S_{base}=10 MVA.
2. Let the base voltage at the generator be U_{1base}=13.8 kV.
3. The transformer ratio gives the base voltage U_{2base}=69 kV for the line and $U_{3base} = 69 \cdot 13.2/66 = 13.8$ kV for the load.

In Figure 5.10, the different sections of the system are given.

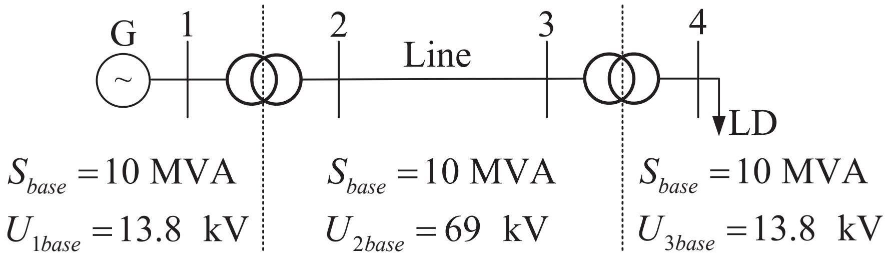


Figure 5.10. Different sections of the system given in Example 5.2.

4. Calculate the per-unit values of the system components.

$$\begin{aligned}
\text{G:}\quad & U_{gpu} = && U_{1pu} = \frac{U_g}{U_{1base}} = \frac{13.8}{13.8} = 1.0\ pu\\
\text{T1:}\quad & \overline{Z}_{t1pu} = && \frac{\overline{Z}_{tp}}{Z_{1base}} = j1.524\,\frac{10}{13.8^2} = j0.080\ pu\\
\text{T2:}\quad & \overline{Z}_{t2pu} = && j\frac{8}{100}\frac{13.2^2}{5}\frac{1}{Z_{3base}} = j\frac{8}{100}\frac{13.2^2}{5}\frac{10}{13.8^2} = 0.1464\ pu\\
\text{Line:}\quad & \overline{Z}_{23pu} = && \frac{\mathcal{L}\,\overline{z}_{23}}{Z_{2base}} = 10 \cdot j0.8\,\frac{10}{69^2} = j0.0168\ pu
\end{aligned}$$

$$
\begin{aligned}
\text{LD:} \quad \overline{Z}_{LD} = & \frac{U_n^2}{\overline{S}_{LD}^*} = \frac{U_n^2}{S_{LD}(\cos\phi - j\sin\phi)} = \frac{U_n^2}{S_{LD}}(\cos\phi + j\sin\phi) = \\
= & \frac{13.2^2}{4/0.8}(0.8 + j0.6) = 27.8784 + j20.9088\ \Omega \\
\overline{Z}_{LDpu} = & \frac{\overline{Z}_{LD}}{Z_{3base}} = (27.8784 + j20.9088) \cdot \frac{10}{13.8^2} = 1.4639 + j1.0979\ pu
\end{aligned}
$$

5. By using these values, an impedance diagram can be drawn as shown in Figure 5.11.

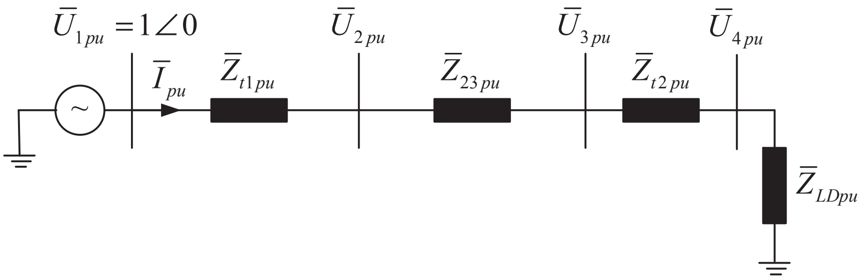


Figure 5.11. Impedance network in per-unit.

6. The current through the network can be calculated as

$$\overline{I}_{pu} = \frac{1 + j0}{j0.08 + j0.0168 + j0.1464 + 1.4639 + j1.0979} = 0.5037\angle{-42.4933^\circ}\ pu \tag{5.18}$$

The load voltage is

$$\overline{U}_{4pu} = \overline{U}_{LDpu} = \overline{I}_{pu}\,\overline{Z}_{LDpu} = 0.9217\angle{-5.6221^\circ}\ pu \tag{5.19}$$

The load power is

$$\overline{S}_{LDpu} = \overline{U}_{LDpu}\overline{I}_{pu}^* = 0.3714 + j0.2785\ pu \tag{5.20}$$

7. The load voltage and active load power in physical units can be obtained by multiplying the per-unit values with corresponding base quantities.

$$U_{LD} = U_{LDpu}\,U_{3base} = 0.9217 \cdot 13.8 = 12.7199\ kV \tag{5.21}$$

$$P_{LD} = Real(\overline{S}_{LDpu})\,S_{base} = 0.3714 \cdot 10 = 3.714\ MW \tag{5.22}$$

Note that the P_{LD} given in the system data (i.e. P_{LD}=4 MW) is the consumed active power at the rated (or nominal) voltage U_n=13.2 kV. However, the actual voltage at bus 4 is 12.7199 kV. Therefore, the actual consumed power is 3.714 MW.

Chapter 6

Power transmission to impedance loads

Transmission lines and cables are normally operating in balanced (or symmetrical) conditions, and as shown in Figure 5.8 a three-phase transmission line (or cable) can be represented with a single-phase line equivalent (or more precisely, with a positive-sequence network, see chapter 8.2). This equivalent can be described by a twoport.

6.1 Twoport theory

Assume that a linear, reciprocal twoport is of interest, where the voltage and current in one end are $\overline{U}_k$ and $\overline{I}_k$ whereas the voltage and current in the other end are $\overline{U}_j$ and $\overline{I}_j$. The conditions valid for this twoport can be described by constants $ABCD$ as

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.1}$$

Assume that the twoport is shortened in the receiving end, (i.e. $\overline{U}_j = 0$) according to Figure 6.1, and that the voltage $\overline{U}$ is applied to the sending end.

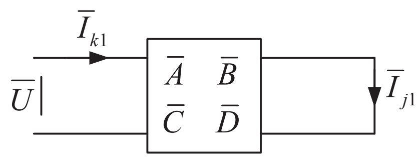


Figure 6.1. Twoport, shortened in the receiving end

For the system shown in Figure 6.1, we have

$$\overline{U} = \overline{A} \cdot 0 + \overline{B} \cdot \overline{I}_{j1} = \overline{B} \cdot \overline{I}_{j1} \tag{6.2}$$
$$\overline{I}_{k1} = \overline{C} \cdot 0 + \overline{D} \cdot \overline{I}_{j1} = \overline{D} \cdot \overline{I}_{j1} \tag{6.3}$$

If it is assumed that the twoport is shortened in the sending end instead, ($\overline{U}_k = 0$) as shown in Figure 6.2, and the voltage $\overline{U}$ is applied to the receiving end. Then according to Figure

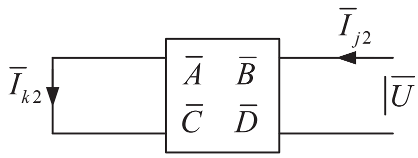


Figure 6.2. Twoport, shortened in the sending end

6.2, we have

$$0 = \overline{A} \cdot \overline{U} - \overline{B} \cdot \overline{I}_{j2} \tag{6.4}$$
$$-\overline{I}_{k2} = \overline{C} \cdot \overline{U} - \overline{D} \cdot \overline{I}_{j2} \tag{6.5}$$

The reciprocity theorem gives that

$$\overline{I}_{k2} = \overline{I}_{j1} = \overline{I} \tag{6.6}$$

From the equations given above, the following expressions can be derived :

$$\text{eq. (6.4)} \Rightarrow \overline{I}_{j2} = \frac{\overline{A}}{\overline{B}}\overline{U} \tag{6.7}$$
$$\text{eq. (6.7)+(6.6)+(6.2)} \Rightarrow \overline{I}_{j2} = \overline{A} \cdot \overline{I} \tag{6.8}$$
$$\text{eq. (6.2)+(6.5)+(6.8)} \Rightarrow -\overline{I} = \overline{C} \cdot \overline{B} \cdot \overline{I} - \overline{D} \cdot \overline{A} \cdot \overline{I} \tag{6.9}$$
$$\text{eq. (6.9)}, \overline{I} \neq 0 \Rightarrow \overline{A} \cdot \overline{D} - \overline{B} \cdot \overline{C} = 1 \tag{6.10}$$

i.e. the determinant of a reciprocal twoport is equal to 1. This implies that if several reciprocal twoports are connected after one another, the determinant of the total twoport obtained is also equal to 1. With three reciprocal twoports F_1, F_2 and F_3 connected after one another, the following is always valid :

$$\det(F_1 F_2 F_3) = \det(F_1)\det(F_2)\det(F_3) = 1 \cdot 1 \cdot 1 = 1 \tag{6.11}$$

6.1.1 Symmetrical twoports

Assume that a symmetrical linear reciprocal twoport is of interest. If the definitions of directions given in Figure 6.3 is used, a current injected in the sending end $\overline{I}_k$ at the voltage

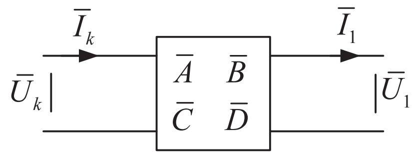


Figure 6.3. Symmetrical twoport, connection 1

$\overline{U}_k$ gives rise to a current $\overline{I}_1$ at the voltage $\overline{U}_1$ in the receiving end. This can be written in an equation as

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{bmatrix} \tag{6.12}$$

Suppose that the circuit is fed in the opposite direction, i.e. $\overline{U}_1$ and $\overline{I}_1$ are obtained in the sending end according to Figure 6.4. This connection can mathematically be formulated as :

$$\begin{bmatrix} \overline{U}_1 \\ -\overline{I}_1 \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ -\overline{I}_j \end{bmatrix} \tag{6.13}$$

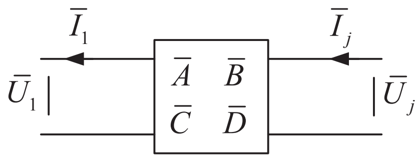


Figure 6.4. Symmetrical twoport, connection 2

By changing the position of the minus sign inside the matrix, equation (6.13) can be rewritten as

$$\begin{bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{bmatrix} = \begin{bmatrix} \overline{A} & -\overline{B} \\ -\overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.14}$$

The matrix in equation (6.14) can be inverted which gives that

$$\begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} = \frac{1}{\underbrace{\overline{A}\cdot\overline{D} - \overline{B}\cdot\overline{C}}_{=1}} \begin{bmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{bmatrix} \tag{6.15}$$

Since the twoport is symmetrical, the following is valid

$$\begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \equiv \begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} \tag{6.16}$$

The equations (6.12), (6.15) and (6.16) give together that

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{bmatrix} \tag{6.17}$$

This concludes that for symmetrical twoports $\overline{A} = \overline{D}$.

6.1.2 Application of twoport theory to transmission line and transformer and impedance load

Note that *all variables in this subsection are expressed in (pu).*

Figure 6.5 shows the π-equivalent model of a line.

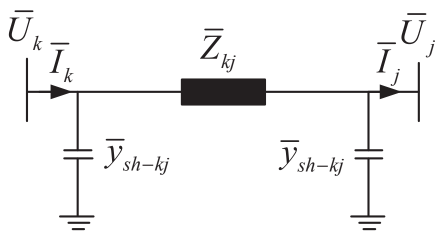


Figure 6.5. π-equivalent model of a line.

From the figure, we have

$$\begin{aligned} \overline{U}_k &= \overline{U}_j + \left(\overline{I}_j + \overline{U}_j \cdot \overline{y}_{sh-kj}\right) \overline{Z}_{kj} \\ \overline{I}_k &= \overline{U}_k \cdot \overline{y}_{sh-kj} + \overline{I}_j + \overline{U}_j \cdot \overline{y}_{sh-kj} \end{aligned} \tag{6.18}$$

These equations can be rewritten as

$$\begin{aligned} \overline{U}_k &= \left(1 + \overline{Z}_{kj} \cdot \overline{y}_{sh-kj}\right) \overline{U}_j + \overline{Z}_{kj} \cdot \overline{I}_j \\ \overline{I}_k &= \overline{y}_{sh-kj} \left(1 + 1 + \overline{Z}_{kj} \cdot \overline{y}_{sh-kj}\right) \overline{U}_j + \left(\overline{Z}_{kj} \cdot \overline{y}_{sh-kj} + 1\right) \overline{I}_j \end{aligned} \tag{6.19}$$

and by using the matrix notation, this can be written as a twoport equation

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} \overbrace{1 + \overline{y}_{sh-kj} \cdot \overline{Z}_{kj}}^{\overline{A}} & \overbrace{\overline{Z}_{kj}}^{\overline{B}} \\ \underbrace{\overline{y}_{sh-kj}(2 + \overline{y}_{sh-kj} \cdot \overline{Z}_{kj})}_{\overline{C}} & \underbrace{1 + \overline{y}_{sh-kj} \cdot \overline{Z}_{kj}}_{\overline{D}} \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.20}$$

As shown in equation (6.20), a line is symmetrical which gives that $\overline{A} = \overline{D}$. A line is also reciprocal which gives that $\overline{A} \cdot \overline{D} - \overline{B} \cdot \overline{C} = 1$.

Using the short line model, then $\overline{y}_{sh-kj} = 0$. Therefore, the twoport equation for a short line model is given by

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} 1 & \overline{Z}_{kj} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.21}$$

The per-unit impedance diagram of a transformer is similar to the per-unit impedance diagram of a short line. Therefore, the twoport equation for a transformer is similar to the twoport equation of a short line model, i.e.

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} 1 & \overline{Z}_t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.22}$$

Figure 6.6 shows the per-unit impedance diagram of an impedance load.

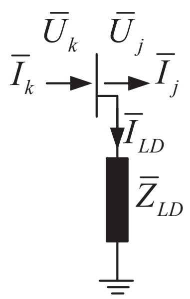


Figure 6.6. Impedance diagram of an impedance load.

From the figure, the following can be easily obtained.

$$\begin{aligned}\overline{U}_k &= \overline{U}_j \\ \overline{I}_k &= \overline{I}_j + \overline{I}_{LD} = \overline{I}_j + \frac{\overline{U}_j}{\overline{Z}_{LD}}\end{aligned} \tag{6.23}$$

Therefore, the twoport equation for an impedance load is given by

$$\begin{bmatrix} \overline{U}_k \\ \overline{I}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\overline{Z}_{LD}} & 1 \end{bmatrix} \begin{bmatrix} \overline{U}_j \\ \overline{I}_j \end{bmatrix} \tag{6.24}$$

6.1.3 Connection to network

As discussed in section 4.5, based on Thévenin-Helmholtz's theorem from any output terminal in a linear electric network the entire linear electric network as seen from the output terminal can be modelled as an ideal voltage source $\overline{U}_{Th}$ in series with an impedance $\overline{Z}_{Th}$. Considering any bus in a linear electric network as an output terminal, as seen from any bus k the network can be replaced with a Thévenin equivalent as shown in Figure 6.7, where $\overline{U}_k = \overline{U}_{Th}$. Assume that a solid three-phase short circuit (i.e. $\overline{Z}_k = 0$) is applied to bus k.

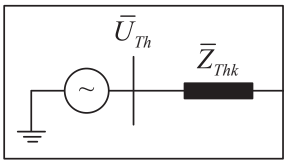


Figure 6.7. Thévenin equivalent of the network as seen from bus k.

This model implies that the short circuit current is

$$\overline{I}_{sck} = \frac{\overline{U}_{Th}\,\text{kV}}{\sqrt{3}\,\overline{Z}_{Thk}\,\Omega}\,\text{kA} \quad \text{or} \quad \overline{I}_{sck} = \frac{\overline{U}_{Th}\,\text{p.u}}{\overline{Z}_{Thk}\,\text{p.u}}\,\text{p.u} \tag{6.25}$$

The question is now how well this model can be adapted to real conditions. For instance, consider the simple system shown in Figure 6.8, where LD is an impedance load and short line model is used for the lines.

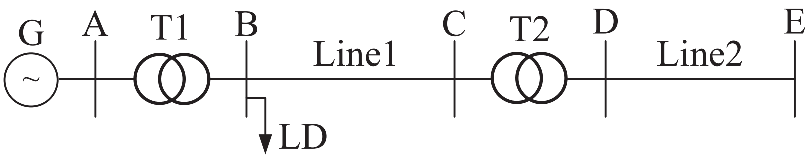


Figure 6.8. A simple system.

Assume that the initial voltage at bus D is known, i.e. $\overline{U}_{Di} = U_{Di}\angle\theta_{Di}$ (p.u). If the pu-values of all components are known, then as seen from bus D the following Thévenin equivalent can be obtained,

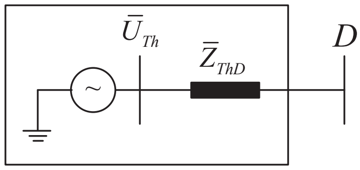


Figure 6.9. Thévenin equivalent seen from bus D.

where,

$$\overline{U}_{Th} = \overline{U}_{Di} \quad \text{and} \quad \overline{Z}_{ThD} = \frac{\overline{Z}_{t1}\,\overline{Z}_{LD}}{\overline{Z}_{t1} + \overline{Z}_{LD}} + \overline{Z}_{BC} + \overline{Z}_{t2}$$

If the pu-values of all components are not known, by applying a solid three-phase short circuit to bus D, the short circuit current can be measured and converted to per unit (i.e. $\overline{I}_{sck}$ (p.u) will be known). Then, the Thévenin impedance as seen from bus D can be calculated as

$$\overline{Z}_{ThD} = \frac{\overline{U}_{Th}}{\overline{I}_{sck}} \quad \text{p.u}$$

Having connected an impedance load $\overline{Z}_{LDD}$ (p.u) to bus D, the voltage at bus D will be

$$\overline{U}_D = \frac{\overline{Z}_{LDD}}{\overline{Z}_{ThD} + \overline{Z}_{LDD}}\overline{U}_{Th} \quad \Rightarrow \quad U_D = \left|\frac{\overline{Z}_{LDD}}{\overline{Z}_{ThD} + \overline{Z}_{LDD}}\right| U_{Th} \tag{6.26}$$

i.e., the voltage magnitude at bus D will drop with

$$\left|\frac{\overline{Z}_{LDD}}{\overline{Z}_{ThD} + \overline{Z}_{LDD}}\right| .100\,\%$$

Now assume that the transformer T2 has a regulator to automatically regulate the voltage magnitude at bus D to its initial value, i.e. U_{Di} (p.u). This kind of transformer is known as On Load Tap Changer (OLTC). When the load is connected to bus D, the OLTC regulates the voltage at bus D to U_{Di}, i.e. $U_D = U_{Di}$ not the voltage given in equation (6.26). The Thévenin equivalent is not valid in this case. Thévenin-Helmholtz's theorem is applied to linear circuits with passive components (static linear circuits), and an OLTC is not a passive component.

Next, as seen from bus E the following Thévenin equivalent can be obtained,

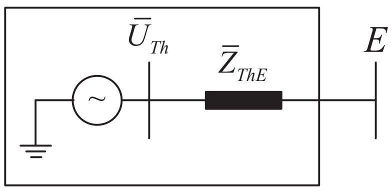


Figure 6.10. Thévenin equivalent as seen from bus E.

where, $\overline{Z}_{ThE} = \overline{Z}_{ThD} + \overline{Z}_{DE}$ and $\overline{U}_{Th} = \overline{U}_{Ei}$.

Having connected an impedance load $\overline{Z}_{LDE}$ (p.u) to bus E, the voltage at bus E will be

$$\overline{U}_E = \frac{\overline{Z}_{LDE}}{\overline{Z}_{ThE} + \overline{Z}_{LDE}} \overline{U}_{Th} \quad \Rightarrow \quad U_E = \left| \frac{\overline{Z}_{LDE}}{\overline{Z}_{ThE} + \overline{Z}_{LDE}} \right| U_{Th} \tag{6.27}$$

i.e., the voltage magnitude at bus E will drop with

$$\left| \frac{\overline{Z}_{LDE}}{\overline{Z}_{ThE} + \overline{Z}_{LDE}} \right| .100\,\%$$

If transformer T2 is an OLTC, the voltage at bus D will be recovered to its initial value (i.e. $U_D = U_{Di}$), but not the voltage at bus E. Therefore, the voltage at bus E when the load i connected will be

$$U_E = \left| \frac{\overline{Z}_{LDE}}{\overline{Z}_{DE} + \overline{Z}_{LDE}} \right| U_{Di} = \left| \frac{\overline{Z}_{LDE}}{(\overline{Z}_{ThE} - \overline{Z}_{ThD}) + \overline{Z}_{LDE}} \right| U_{Di} \tag{6.28}$$

The conclusion is that the equivalent impedance from a bus located out in a distribution system (with a fairly weak voltage) to the closest bus with regulated voltage can be calculated as the difference between the Thévenin impedance from the bus with weak voltage and the Thévenin impedance from the bus with voltage regulation. To calculate the voltage drop at the connection of the load, the calculated equivalent impedance and the voltage at the regulated bus will be used in the Thévenin equivalent model.

In some cases, the term short circuit capacity $\overline{S}_{sck}$ at a bus k is used. It is defined as

$$\overline{S}_{sck} = \overline{U}_{Th} \overline{I}^*_{sck} = U_{Th}\, I_{sck} \angle \phi_{sck} \quad \text{p.u} \tag{6.29}$$

which gives the power that is obtained in the Thévenin impedance. Since this impedance often is mostly reactive we have $\phi_{sck} \approx 90°$. The short circuit capacity is of interest when the loadability of a certain bus is concerned. The short circuit capacity indicates how much the bus voltage will change for different loading at that bus. The voltage increase at generator buses can be also calculated.

Example 6.1 *At a bus with a pure inductive short circuit capacity of 500 MVA (i.e.* $\cos\phi_{sck} = 0$*) an impedance load of 4 MW,* $\cos\phi_{LD} = 0.8$ *at nominal voltage, is connected. Calculate the change in the bus voltage when the load is connected.*

Solution

Assume a voltage of 1 pu and a base power S_{base} = 500 MVA, i.e. $\overline{S}_{scpu} = 1\angle 90°$. The network can then be modeled as shown in Figure 6.11.

The Thévenin impedance can be calculated according to equation (6.25) and (6.29) :

$$\overline{Z}_{Thpu} = \frac{\overline{U}_{Thpu}}{\overline{I}_{scpu}} = \frac{U^2_{Thpu}}{\overline{S}^*_{scpu}} = \frac{1}{1\angle -90°} = j1 \tag{6.30}$$

The load impedance can be calculated as

$$\overline{Z}_{LDpu} = \frac{U^2_{npu}}{\overline{S}^*_{LDpu}} = \frac{U^2_{npu}}{\frac{P_{LD}}{S_{base}\cdot\cos\phi_{LD}}} (\cos\phi_{LD} + j\sin\phi_{LD}) = \frac{1^2}{\frac{4}{500\cdot 0.8}} (0.8 + j0.6) \tag{6.31}$$

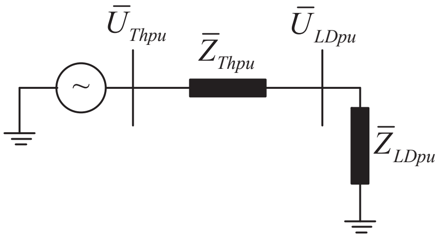


Figure 6.11. Single-phase model of system given in the example.

Thus, the voltage $\overline{U}_{LD}$ at the load is

$$\overline{U}_{LDpu} = \frac{\overline{Z}_{LDpu}}{\overline{Z}_{Thpu} + \overline{Z}_{LDpu}} \overline{U}_{Thpu} = \frac{80 + j60}{j1 + 80 + j60} 1\angle 0 = 0.9940\angle - 0.4556° \tag{6.32}$$

i.e. the voltage drop is about 0.6 %.

Conclusion : A load with an apparent power of 1 % of the short circuit capacity at the bus connected, will cause a voltage drop at that bus of ≈ 1 %.

Example 6.2 *As shown in Figure 6.12, a small industry (LD) is fed by a power system via a transformer (5 MVA, 70/10, x = 4 %) which is located at a distance of 5 km. The electric power demand of the industry is 400 kW at cosϕ=0.8, lagging, at a voltage of 10 kV. The industry can be modeled as an impedance load. The 10 kV line has an series impedance of 0.9+j0.3 Ω/km and a shunt admittance of $j3 \times 10^{-6}$ S/km. Assume that the line is modeled by the π-equivalent. When the transformer is disconnected from bus 3, the voltage at this bus is 70 kV, and a three-phase short circuit applied to this bus results in a pure inductive short circuit current of 0.3 kA.*

Calculate the voltage at the industry as well as the power fed by the transformer into the line.

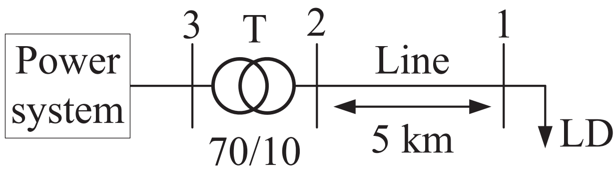


Figure 6.12. Single-line diagram of the system in Example 6.2.

Solution

Choose the base values (MVA, kV, ⇒ kA, Ω) :

$S_{base} = 0.5$ MVA, $U_{base10} = 10$ kV $\Rightarrow I_{base10} = S_{base}/\sqrt{3}U_{base10} = 0.0289$ kA, $Z_{base10} = U_{base10}^2/S_{base} = 200\ \Omega$

$U_{base70} = 70$ kV $\Rightarrow I_{base70} = S_{base}/\sqrt{3}U_{base70} = 0.0041$ kA

Calculate the per-unit values of the Thévenin equivalent of the system:

$$\overline{U}_{Thpu} = \frac{\overline{U}_{Th}}{U_{base70}} = \frac{70\angle 0}{70} = 1\angle 0^\circ = 1 \quad \text{and} \quad \overline{I}_{scpu} = \frac{\overline{I}_{sc}}{I_{base70}} = \frac{0.3\angle -90^\circ}{0.00412} = 72.8155\angle -90^\circ$$

$$\overline{Z}_{Thpu} = \frac{\overline{U}_{Thpu}}{\overline{I}_{scpu}} = j0.0137$$

Calculate the per-unit values of the transformer:

$$\overline{Z}_{tpu} = \frac{\overline{Z}_{t\%}}{100}\frac{Z_{tbase10}}{Z_{base10}} = \frac{\overline{Z}_{t\%}}{100}\frac{U_{2n}^2}{S_{nt}}\frac{S_{base}}{U_{base10}^2} = \frac{j4}{100}\frac{10^2}{5}\frac{0.5}{10^2} = \frac{j4}{100}\frac{0.5}{5} = j0.004$$

Calculate the per-unit values of the line:

$$\begin{aligned}
\overline{Z}_{21pu} &= \frac{5\cdot(0.9+j0.3)}{Z_{base10}} = 0.0225 + j0.0075 \\
\overline{y}_{sh-21pu} &= \frac{\overline{Y}_{sh-21pu}}{2} = \frac{5\cdot(j3\times 10^{-6})}{2} Z_{base10} = \frac{j0.003}{2} \\
\overline{A}_L &= 1 + \overline{y}_{sh-21pu}\cdot\overline{Z}_{21pu} = 1.0000 + j0.0000 \\
\overline{B}_L &= \overline{Z}_{21pu} = 0.0225 + j0.0075 \\
\overline{C}_L &= \overline{y}_{sh-21pu}(2 + \overline{y}_{sh-21pu}\cdot\overline{Z}_{21pu}) = 0.0000 + j0.0030 \\
\overline{D}_L &= \overline{A}_L = 1.0000 + j0.0000
\end{aligned}$$

Calculate the per-unit values of the industry impedance:

$$\overline{Z}_{LDpu} = \frac{U_n^2}{\overline{S}_{LD}^*}\frac{1}{Z_{base10}} = \frac{10^2}{\frac{0.4}{0.8}}(0.8+j0.6)\frac{1}{200} = 0.8 + j0.6$$

Figure 6.13 shows the per-unit impedance diagram of the entire system, where the power system has been modelled by its Thévenin equivalent. Bus 4 (the terminal bus of the ideal voltage source) is termed as infinite bus.

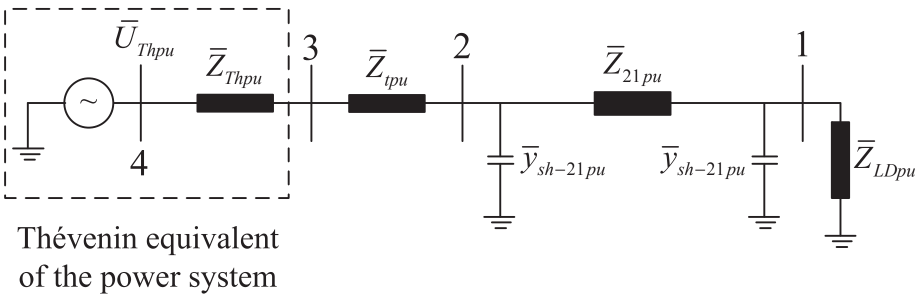


Figure 6.13. Per-unit impedance diagram of the system in Example 6.2.

The twoport of the above system (from the infinite bus to bus 1) can be formulated as

$$\begin{bmatrix}\overline{U}_{Thpu}\\ \overline{I}_{4pu}\end{bmatrix} = \begin{bmatrix}1 & \overline{Z}_{Thpu}+\overline{Z}_{tpu}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overline{U}_{2pu}\\ \overline{I}_{2pu}\end{bmatrix} = \begin{bmatrix}1 & \overline{Z}_{Thpu}+\overline{Z}_{tpu}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overline{A}_L & \overline{B}_L\\ \overline{C}_L & \overline{D}_L\end{bmatrix}\begin{bmatrix}\overline{U}_{1pu}\\ \overline{I}_{1pu}\end{bmatrix} =$$

$$= \begin{bmatrix}\overline{A} & \overline{B}\\ \overline{C} & \overline{D}\end{bmatrix}\begin{bmatrix}\overline{U}_{1pu}\\ \overline{I}_{1pu}\end{bmatrix} = \begin{bmatrix}0.9999+j0.0000 & 0.0225+j0.0252\\ 0.0000+j0.0030 & 1.0000+j0.0000\end{bmatrix}\begin{bmatrix}\overline{U}_{1pu}\\ \overline{I}_{1pu}\end{bmatrix}$$

As seen from bus 4, the impedance of the entire system (including the industry) can be calculated as

$$
\begin{aligned}
\overline{Z}_{totpu} &= \frac{\overline{U}_{Thpu}}{\overline{I}_{4pu}} = \frac{\overline{A}\,\overline{U}_{1pu} + \overline{B}\,\overline{I}_{1pu}}{\overline{C}\,\overline{U}_{1pu} + \overline{D}\,\overline{I}_{1pu}} = \frac{\overline{A}\,\frac{\overline{U}_{1pu}}{\overline{I}_{1pu}} + \overline{B}}{\overline{C}\,\frac{\overline{U}_{1pu}}{\overline{I}_{1pu}} + \overline{D}} = \frac{\overline{A}\,\overline{Z}_{LDpu} + \overline{B}}{\overline{C}\,\overline{Z}_{LDpu} + \overline{D}} = 0.8254 + j0.6244 \\
&\Rightarrow \overline{I}_{4pu} = \overline{U}_{Thpu}/\overline{Z}_{totpu} = 0.9662\angle - 37.1035^\circ
\end{aligned}
$$

The power fed by the transformer into the line can be calculated as

$$
\begin{aligned}
\begin{bmatrix} \overline{U}_{2pu} \\ \overline{I}_{2pu} \end{bmatrix} &= \begin{bmatrix} 1 & \overline{Z}_{Thpu} + \overline{Z}_{trapu} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_{Thpu} \\ \overline{I}_{4pu} \end{bmatrix} = \begin{bmatrix} 0.9898\angle - 0.7917^\circ \\ 0.9662\angle - 37.1035^\circ \end{bmatrix} \\
&\Rightarrow \overline{S}_2 = \overline{U}_{2pu}\,\overline{I}^{*}_{2pu}\,S_{base} = 0.3853 + j0.2832 \text{ MVA}
\end{aligned}
$$

the voltage at the industry can be calculated as

$$
\begin{aligned}
\begin{bmatrix} \overline{U}_{1pu} \\ \overline{I}_{1pu} \end{bmatrix} &= \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix}^{-1} \begin{bmatrix} \overline{U}_{Thpu} \\ \overline{I}_{4pu} \end{bmatrix} = \begin{bmatrix} 0.9680\angle - 0.3733^\circ \\ 0.9680\angle - 37.2432^\circ \end{bmatrix} \\
&\Rightarrow U_1 \text{ (kV)} = U_{1pu}\,U_{base10} = 9.6796 \text{ kV}
\end{aligned}
$$

6.2 A general method for analysis of linear balanced three-phase systems

When analyzing large power systems, it is necessary to perform the analysis in a systematic manner. Below, a small system is analyzed with a method which can be used for large systems. In Figure 6.14, an impedance load $\overline{Z}_{LD1}$ is fed from an infinite bus (i.e. bus 3

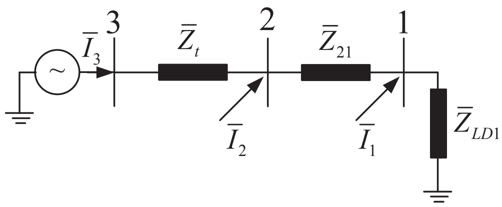


Figure 6.14. Per unit impedance diagram of a balanced power system.

which is the terminal bus of the ideal voltage source) via a transformer with impedance $\overline{Z}_t$ and a line with impedance $\overline{Z}_{21}$. The voltage at the infinite bus is $\overline{U}_3$. All variables are expressed in per unit. The Y-bus matrix for this system can be formulated as

$$
\begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{bmatrix} = \mathbf{I} = \mathbf{YU} = \begin{bmatrix} \frac{1}{\overline{Z}_{LD1}} + \frac{1}{\overline{Z}_{21}} & -\frac{1}{\overline{Z}_{21}} & 0 \\ -\frac{1}{\overline{Z}_{21}} & \frac{1}{\overline{Z}_{21}} + \frac{1}{\overline{Z}_t} & -\frac{1}{\overline{Z}_t} \\ 0 & -\frac{1}{\overline{Z}_t} & \frac{1}{\overline{Z}_t} \end{bmatrix} \begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \end{bmatrix} \tag{6.33}
$$

The Y-bus matrix can be inverted which results in the corresponding Z-bus matrix :

$$
\mathbf{U} = \mathbf{Y}^{-1}\mathbf{I} = \mathbf{ZI} \tag{6.34}
$$

Since $\overline{I}_1 = \overline{I}_2 = 0$, the third row in equation (6.34) can be written as

$$\overline{U}_3 = \mathbf{Z}(3,3) \cdot \overline{I}_3 \quad \Rightarrow \quad \overline{I}_3 = \frac{\overline{U}_3}{\mathbf{Z}(3,3)} \tag{6.35}$$

where $\mathbf{Z}(3,3)$ is an element in the Z-bus matrix. With that value of the current inserted into equation (6.34), all system voltages are obtained.

$$\begin{aligned} \overline{U}_1 &= \mathbf{Z}(1,3) \cdot \overline{I}_3 \\ \overline{U}_2 &= \mathbf{Z}(2,3) \cdot \overline{I}_3 \end{aligned} \tag{6.36}$$

Corresponding calculations can be performed for arbitrarily large systems containing impedance loads and one voltage source.

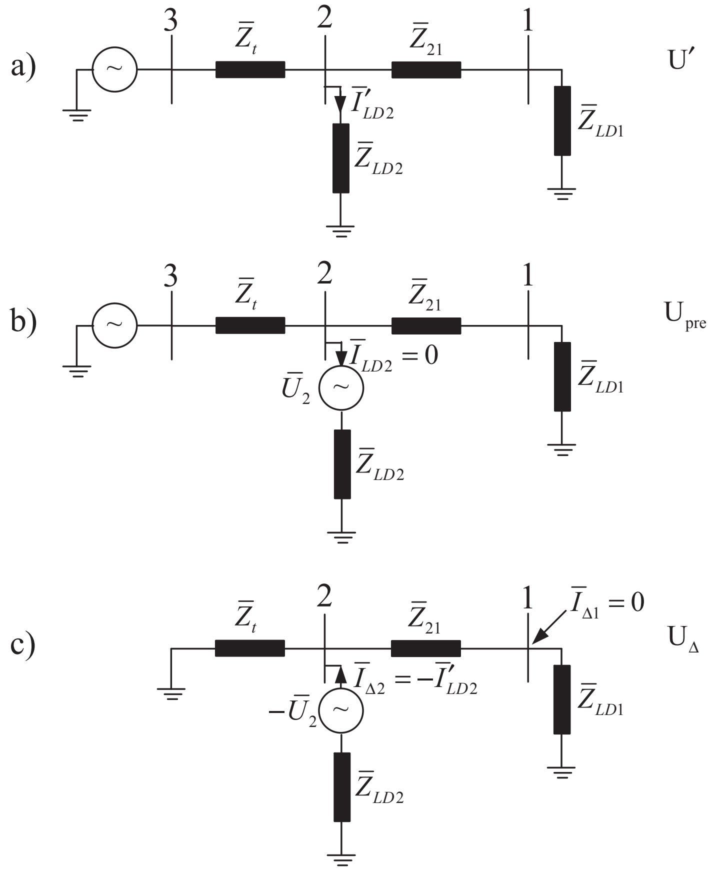


Figure 6.15. Total voltage obtained by using superposition

Assume that an impedance $\overline{Z}_{LD2}$ is added to the system at bus 2, as shown in Figure 6.15 a). This will change the voltage magnitudes at all buses with exception of the bus connected to the voltage source (bus 3 in this example). Then, the actual voltages can be expressed by

$$\mathbf{U'} = \mathbf{U_{pre}} + \mathbf{U_{\Delta}} \tag{6.37}$$

where $\mathbf{U'}$ is a vector containing the actual voltages due to the change, $\mathbf{U_{pre}}$ is a vector containing the voltages of all buses (with exception of the bus connected to the voltage source) prior to the change and $\mathbf{U_\Delta}$ is the applied change. This equation can be illustrated graphically as shown in Figure 6.15, i.e. the total voltage can be calculated as a superposition of two systems with equal impedances but with different voltage sources.

As indicated by the system in Figure 6.15 c), the feeding voltage is $-\overline{U}_2$ while the voltage source at bus 3 is shortened (when the voltage source is short circuited the bus connected to the voltage source (i.e. bus 3) is removed). The Y-bus matrix for this system can be obtained by removing the row and column corresponding to bus 3 in $\mathbf{Y}$ (see equation (6.33)) since bus 3 is grounded and removed. If bus 3 was kept in the mathematical formulation, $\mathbf{Y}(3,3) = \infty$ since the impedance to ground is zero.

$$\begin{bmatrix} \overline{I}_{\Delta 1} \\ \overline{I}_{\Delta 2} \end{bmatrix} = \mathbf{I_\Delta} = \mathbf{Y_\Delta U_\Delta} = \begin{bmatrix} \frac{1}{\overline{Z}_{LD1}} + \frac{1}{\overline{Z}_{21}} & -\frac{1}{\overline{Z}_{21}} \\ -\frac{1}{\overline{Z}_{21}} & \frac{1}{\overline{Z}_{21}} + \frac{1}{\overline{Z}_t} \end{bmatrix} \begin{bmatrix} \overline{U}_{\Delta 1} \\ \overline{U}_{\Delta 2} \end{bmatrix} \tag{6.38}$$

The expression given above, can be inverted which gives the corresponding Z-bus matrix :

$$\mathbf{U_\Delta} = \mathbf{Y_\Delta^{-1} I_\Delta} = \mathbf{Z_\Delta I_\Delta} \quad \Rightarrow \quad \begin{bmatrix} \overline{U}_{\Delta 1} \\ \overline{U}_{\Delta 2} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_\Delta(1,1) & \mathbf{Z}_\Delta(1,2) \\ \mathbf{Z}_\Delta(2,1) & \mathbf{Z}_\Delta(2,2) \end{bmatrix} \begin{bmatrix} \overline{I}_{\Delta 1} \\ \overline{I}_{\Delta 2} \end{bmatrix} \tag{6.39}$$

In this equation, $\overline{I}_{\Delta 1} = 0$ which is shown in Figure 6.15 c). This gives that the second row can be written as

$$\overline{U}_{\Delta 2} = \mathbf{Z}_\Delta(2,2)\overline{I}_{\Delta 2} \tag{6.40}$$

Figure 6.14 gives the same currents as Figure 6.15 b), since the voltage over $\overline{Z}_{LD2}$ in Figure 6.15 b) is zero. This implies that the current through $\overline{Z}_{LD2}$ is zero. Therefore, $\overline{I}_{\Delta 2} = -\overline{I}'_{LD2}$. At bus 2 in Figure 6.15 a) the following is valid

$$\overline{U}'_2 = \overline{I}'_{LD2} \cdot \overline{Z}_{LD2} = -\overline{I}_{\Delta 2} \cdot \overline{Z}_{LD2} \tag{6.41}$$

By combining equations (6.37), (6.40) and (6.41), the following can be obtained

$$\overline{I}_{\Delta 2} = \frac{-\overline{U}_2}{\overline{Z}_{LD2} + \mathbf{Z}_\Delta(2,2)} \tag{6.42}$$

By inserting that value in the equations given above, all voltages after the system change can be calculated as :

$$\overline{U}'_2 = \frac{\overline{Z}_{LD2}}{\overline{Z}_{LD2} + \mathbf{Z}_\Delta(2,2)}\overline{U}_2 \tag{6.43}$$

$$\overline{U}'_1 = \overline{U}_1 - \frac{\mathbf{Z}_\Delta(1,2)}{\overline{Z}_{LD2} + \mathbf{Z}_\Delta(2,2)}\overline{U}_2 \tag{6.44}$$

The procedure given above can be generalized to be used for an arbitrarily large system. Assume that an impedance $\overline{Z}_r$ is connected to a bus r and an arbitrary bus is termed i. The current $\overline{I}'_r$ ($=-\overline{I}_{\Delta r}$) through $\overline{Z}_r$ can be calculated as well as the voltages after connection of the impedance $\overline{Z}_r$ at bus r. The equations are as follows.

$$\overline{I}'_r = \frac{\overline{U}_r}{\overline{Z}_r + \mathbf{Z}_\Delta(r,r)} \tag{6.45}$$

$$\overline{U}'_r = \frac{\overline{Z}_r}{\overline{Z}_r + \mathbf{Z}_\Delta(r,r)}\overline{U}_r \tag{6.46}$$

$$\overline{U}'_i = \overline{U}_i - \frac{\mathbf{Z}_\Delta(i,r)}{\overline{Z}_r + \mathbf{Z}_\Delta(r,r)}\overline{U}_r \tag{6.47}$$

Note that $i \neq r$, and bus r and bus i do not represent the bus connected to the voltage source.

The Thévenin equivalent at a bus in a symmetrical network can be calculated by using equations (6.37) and (6.39). At bus r (r=2 in this case), the equation will be as

$$\mathbf{U}'(r) = \mathbf{U_{pre}}(r) + \mathbf{Z}_\Delta(r,r)\mathbf{I}_\Delta(r) \tag{6.48}$$

where

$\mathbf{U_{pre}}(r) = \overline{U}_{Thr}$ Thévenin voltage at bus r prior to the change, see Figure 6.16.

$\mathbf{Z}_\Delta(r,r) = \overline{Z}_{Thr}$ Thévenin impedance as seen from bus r, see Figure 6.16.

$\mathbf{I}_\Delta(r) = -\overline{I}'_r$ The actual injected current into bus r.

$\mathbf{U}'(r) = \overline{U}'_r$ the actual voltage at bus r.

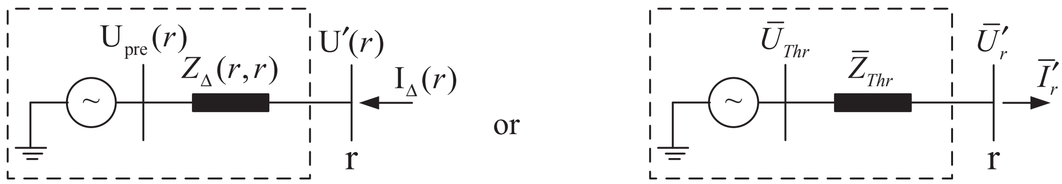


Figure 6.16. Thévenin equivalent at bus r in a symmetrical three-phase network.

As given by equation (6.48) and Figure 6.16, $\mathbf{U}'(r) = \mathbf{U_{pre}}(r)$ if $\mathbf{I}_\Delta(r) = 0$. This formulation shows that the Thévenin voltage at bus r can be calculated as the voltage at bus r when the bus is not loaded, i.e. $\mathbf{I}_\Delta(r) = 0$. The Thévenin impedance is found as the r-th diagonal element of the impedance matrix $\mathbf{Z_\Delta}$ which is determined when the voltage source is shortened.

Example 6.3 *In Figure 6.17, an internal network of an industry is given. Power is delivered by an infinite bus with a nominal voltage at bus* **1**. *Power is transmitted via transformer* $T1$, *Line2 and transformer* $T2$ *to the load* $LD2$. *There is also a high voltage load* $LD1$ *connected*

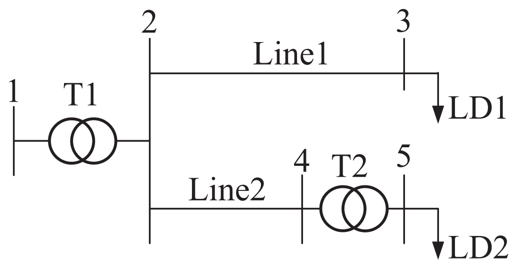


Figure 6.17. Single-line diagram of an internal industry network

to $T1$ *via Line1. The system data is given as follows:*

- *Transformer $T1$: 800 kVA, 70/10, $x = 7$ %*
- *Transformer $T2$: 300 kVA, 10/0.4, $x = 8$ %*
- *Line1 : $r = 0.17\ \Omega/km$, $x = 0.3\Omega/km$, $b_c = 3.2 \times 10^{-6}\ S/km$, $\mathcal{L} = 2\ km$*
- *Line2 : $r = 0.17\ \Omega/km$, $x = 0.3\Omega/km$, $b_c = 3.2 \times 10^{-6}\ S/km$, $\mathcal{L} = 1\ km$*
- *Load $LD1$: impedance load, 500 kW, cosϕ = 0.80, inductive at 10 kV*
- *Load $LD2$: impedance load, 200 kW, cosϕ = 0.95, inductive at 0.4 kV*

The π-equivalent model is used for the lines.

Calculate the efficiency of the internal network as well as the short circuit current that is obtained at a solid three-phase short circuit at bus 4.

Solution

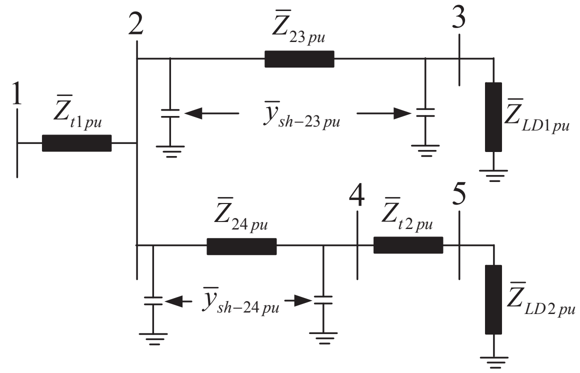


Figure 6.18. Network in Example 6.3.

Choose the base values (MVA, kV, ⇒ kA, Ω) :
$S_{base} = 500$ kVA $= 0.5$ MVA, $U_{base70} = 70$ kV

$U_{base10} = 10$ kV $\Rightarrow I_{base10} = S_{base}/\sqrt{3}U_{base10} = 0.0289$ kA, $Z_{base10} = U^2_{base10}/S_{base} = 200\ \Omega$

$U_{base04} = 0.4$ kV $\Rightarrow I_{base04} = S_{base}/\sqrt{3}U_{base04} = 0.7217$ kA, $Z_{base04} = U^2_{base04}/S_{base} = 0.32\ \Omega$

Calculate the per-unit values of the infinite bus :
$\overline{U}_1 = 70/U_{base70} = 70/70 = 1$

Calculate the per-unit values of the transformer $T1$:
$\overline{Z}_{t1pu} = (\overline{Z}_{t1\%}/100) \cdot Z_{t1base10}/Z_{base10} = (\overline{Z}_{T1\%}/100) \cdot S_{base}/S_{nt1} = (j7/100) \cdot 0.5/0.8 = j0.0438$

Calculate the per-unit values of the transformer $T2$:
$\overline{Z}_{t2pu} = (\overline{Z}_{t2\%}/100) \cdot S_{base}/S_{nt2} = (j8/100) \cdot 0.5/0.3 = j0.1333$

Calculate the per-unit values of Line1 :

$\overline{Z}_{23pu} = \mathcal{L}(r + jx)/Z_{base10} = 2 \cdot (0.17 + j0.3)/Z_{base10} = 0.0017 + j0.003$
$\overline{y}_{sh-23pu} = j\mathcal{L}\, b_c\, Z_{base10}/2 = j2 \cdot (3.2 \times 10^{-6}) \cdot Z_{base10}/2 = j0.0013/2$

Calculate the per-unit values of Line2 :
$\overline{Z}_{24pu} = \mathcal{L}(r + jx)/Z_{base10} = 1 \cdot (0.17 + j0.3)/Z_{base10} = 0.0009 + j0.0015$

$\overline{y}_{sh-24pu} = j\mathcal{L}\, b_c\, Z_{base10}/2 = j1 \cdot (3.2 \times 10^{-6}) \cdot Z_{base10}/2 = j0.00064/2$
Calculate the per-unit values of the impedance $LD1$:
$\overline{Z}_{LD1pu} = (U_{LD1}^2/\overline{S}_{LD1}^*)/Z_{base10} = (10^2/[0.5/0.8]) \cdot (0.8 + j0.6)/200 = 0.64 + j0.48$

Calculate the per-unit values of the impedance $LD2$:
$\overline{Z}_{LD2pu} = (U_{LD2}^2/\overline{S}_{LD2}^*)/Z_{base04} = (0.4^2/0.2/0.95) \cdot (0.95 + j\sqrt{1 - 0.95^2})/0.32 = 2.2562 + j0.7416$

Calculate the Y-bus matrix of the network. The grounding point is not included in the Y-bus matrix since the system then is overdetermined.

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{\overline{Z}_{t1pu}} & -\frac{1}{\overline{Z}_{t1pu}} & 0 & 0 & 0 \\ -\frac{1}{\overline{Z}_{t1pu}} & \overline{Y}_{22} & -\frac{1}{\overline{Z}_{23pu}} & -\frac{1}{\overline{Z}_{24pu}} & 0 \\ 0 & -\frac{1}{\overline{Z}_{23pu}} & \overline{Y}_{33} & 0 & 0 \\ 0 & -\frac{1}{\overline{Z}_{24pu}} & 0 & \overline{Y}_{44} & -\frac{1}{\overline{Z}_{t2pu}} \\ 0 & 0 & 0 & -\frac{1}{\overline{Z}_{t2pu}} & \frac{1}{\overline{Z}_{t2pu}} + \frac{1}{\overline{Z}_{LD2pu}} \end{bmatrix} \tag{6.49}$$

where

$$\begin{aligned} \overline{Y}_{22} &= \frac{1}{\overline{Z}_{t1pu}} + \frac{1}{\overline{Z}_{23pu}} + \overline{y}_{sh-23pu} + \frac{1}{\overline{Z}_{24pu}} + \overline{y}_{sh-24pu} \\ \overline{Y}_{33} &= \frac{1}{\overline{Z}_{23pu}} + \overline{y}_{sh-23pu} + \frac{1}{\overline{Z}_{LD1pu}} \\ \overline{Y}_{44} &= \frac{1}{\overline{Z}_{24pu}} + \overline{y}_{sh-24pu} + \frac{1}{\overline{Z}_{t2pu}} \end{aligned}$$

Next, we have

$$\mathbf{I} = \mathbf{YU} \tag{6.50}$$

which can be rewritten as

$$\begin{bmatrix} \overline{U}_1 \\ \overline{U}_2 \\ \overline{U}_3 \\ \overline{U}_4 \\ \overline{U}_5 \end{bmatrix} = \mathbf{U} = \mathbf{Y^{-1}I} = \mathbf{ZI} = \mathbf{Z} \begin{bmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \\ \overline{I}_4 \\ \overline{I}_5 \end{bmatrix} \tag{6.51}$$

The Z-bus matrix can be calculated by inverting the Y-bus matrix :

$$\mathbf{Z} = \begin{bmatrix} 0.510{+}j0.375 & 0.510{+}j0.331 & 0.508{+}j0.329 & 0.510{+}j0.331 & 0.516{+}j0.298 \\ 0.510{+}j0.331 & 0.510{+}j0.331 & 0.508{+}j0.329 & 0.510{+}j0.331 & 0.516{+}j0.298 \\ 0.508{+}j0.329 & 0.508{+}j0.329 & 0.509{+}j0.330 & 0.508{+}j0.329 & 0.515{+}j0.296 \\ 0.510{+}j0.331 & 0.510{+}j0.331 & 0.508{+}j0.329 & 0.510{+}j0.332 & 0.517{+}j0.299 \\ 0.516{+}j0.298 & 0.516{+}j0.298 & 0.515{+}j0.296 & 0.517{+}j0.299 & 0.529{+}j0.397 \end{bmatrix} \tag{6.52}$$

Since all injected currents with exception of $\overline{I}_1$ are zero, $\overline{I}_1$ can be calculated using the first row in equation (6.51) :

$$\overline{U}_1 = \mathbf{Z}(1,1)\overline{I}_1 \Rightarrow \overline{I}_1 = \overline{U}_1/\mathbf{Z}(1,1) = 1.0/(0.510 + j0.375) = 1.58\angle - 36.33^\circ \tag{6.53}$$

The voltages at the other buses can be easily be solved by using equation (6.51) :

$$\begin{aligned}
\overline{U}_2 &= \mathbf{Z}(2,1)\overline{I}_1 = (0.510 + j0.331) \cdot (1.58\angle - 36.33^\circ) = 0.9606\angle - 3.324^\circ \\
\overline{U}_3 &= \mathbf{Z}(3,1)\overline{I}_1 = (0.508 + j0.329) \cdot (1.58\angle - 36.33^\circ) = 0.9569\angle - 3.423^\circ \\
\overline{U}_4 &= \mathbf{Z}(4,1)\overline{I}_1 = (0.510 + j0.331) \cdot (1.58\angle - 36.33^\circ) = 0.9601\angle - 3.350^\circ \\
\overline{U}_5 &= \mathbf{Z}(5,1)\overline{I}_1 = (0.516 + j0.298) \cdot (1.58\angle - 36.33^\circ) = 0.9423\angle - 6.351^\circ
\end{aligned} \tag{6.54}$$

The total amount of power delivered to the industry is

$$\overline{S}_1 = \overline{U}_1 \cdot \overline{I}_1^* \cdot S_{base} = 0.6367 + j0.4682 \text{ MVA} \tag{6.55}$$

The power losses in Line1 and Line2 can be calculated as

$$\begin{aligned}
\overline{I}_{Z23} =& (\overline{U}_2 - \overline{U}_3)/\overline{Z}_{23pu} = 1.1957\angle - 40.27^\circ \\
\overline{I}_{Z24} =& (\overline{U}_2 - \overline{U}_4)/\overline{Z}_{24pu} = 0.3966\angle - 24.50^\circ \\
P_{fLine1} =& Real(\overline{Z}_{23pu}) I_{Z23}^2 \cdot S_{base} = 0.0012 \text{ MW} \\
P_{fLine2} =& Real(\overline{Z}_{24pu}) I_{Z24}^2 \cdot S_{base} = 0.0000669\text{MW}
\end{aligned} \tag{6.56}$$

The efficiency for the network is then

$$\eta = \frac{Real(\overline{S}_1) - P_{fLine1} - P_{fLine2}}{Real(\overline{S}_1)} = 0.9980 \Rightarrow 99.80\% \tag{6.57}$$

A solid short circuit at bus 4 can be calculated by connecting an impedance with $\overline{Z}_4 = 0$ at bus 4. According to section 6.2, the current through the impedance $\overline{Z}_4$ can be determined by removing the row and the column of the Y-bus matrix that corresponds to the bus connected to the voltage source (i.e. bus 1 in this example). Thus,

$$\mathbf{Y_\Delta} = \mathbf{Y}(2:5, 2:5) = \begin{bmatrix} \overline{Y}_{22} & -\frac{1}{\overline{Z}_{23pu}} & -\frac{1}{\overline{Z}_{24pu}} & 0 \\ -\frac{1}{\overline{Z}_{23pu}} & \overline{Y}_{33} & 0 & 0 \\ -\frac{1}{\overline{Z}_{24pu}} & 0 & \overline{Y}_{44} & -\frac{1}{\overline{Z}_{t2pu}} \\ 0 & 0 & -\frac{1}{\overline{Z}_{t2pu}} & \frac{1}{\overline{Z}_{t2pu}} + \frac{1}{\overline{Z}_{LD2pu}} \end{bmatrix} \tag{6.58}$$

The inverse of this matrix is

$$\mathbf{Z_\Delta} = \mathbf{Y_\Delta^{-1}} = \begin{bmatrix} 0.0024+j0.0420 & 0.0025+j0.0418 & 0.0025+j0.0419 & 0.0046+j0.0410 \\ 0.0025+j0.0418 & 0.0043+j0.0446 & 0.0025+j0.0418 & 0.0046+j0.0408 \\ 0.0025+j0.0419 & 0.0025+j0.0418 & 0.0033+j0.0434 & 0.0055+j0.0424 \\ 0.0046+j0.0410 & 0.0046+j0.0408 & 0.0055+j0.0424 & 0.0144+j0.1719 \end{bmatrix} \tag{6.59}$$

The short circuit current at bus 4 can then be calculated according to equation (6.45).

$$\begin{aligned}
\overline{I}_{sc4} &= \frac{\overline{U}_4}{\overline{Z}_4 + \underbrace{\mathbf{Z}_\Delta(4,4)}_{\text{element (3,3) in } \mathbf{z_\Delta}}} I_{base10} = \frac{0.9601\angle - 3.350^\circ}{0 + (0.0033 + j0.0434)} 0.0289 = \\
&= 0.6366\angle - 88.97^\circ \text{ kA}
\end{aligned} \tag{6.60}$$

6.3 Extended method to be used for power loads

The method described in section 6.2 is valid when all system loads are modeled as impedance loads, i.e. the power consumed is proportional to the voltage squared. In steady-state conditions, an often used load model is the constant power model. The method described in section 6.2 can be used in an iterative way, described as follows:

1. Calculate the per-unit values of all components that are of interest. Loads that are modeled with constant power (independent of the voltage) are replaced by impedances. The impedance of a load at bus k can be calculated as $\overline{Z}_{LDk} = U_n^2/\overline{S}^*_{LDk}$ where $U_n = 1$ pu is the rated (or nominal) voltage, and $\overline{S}_{LDk}$ is the rated power of the load.

2. Calculate the Y-bus matrix and the corresponding Z-bus matrix of the network as well as the load impedances. By using the method described in section 6.2 (equation (6.35) and (6.36)), the voltage at all buses can be calculated.

3. Calculate the load demand at all loads. The power demand $\overline{S}_{LDk-b}$ at load LDk is obtained as $\overline{S}_{LDk-b} = U_k^2/\overline{Z}^*_{LDk}$ where U_k is the actual calculated voltage at bus k.

4. Calculate the difference between the actual calculated and specified load demand for all power loads :

$$\Delta P_{LDk} = |Re(\overline{S}_{LDk-b}) - Re(\overline{S}_{LDk})| \tag{6.61}$$
$$\Delta Q_{LDk} = |Im(\overline{S}_{LDk-b}) - Im(\overline{S}_{LDk})| \tag{6.62}$$

5. If ΔP_{LDk} and/or ΔQ_{LDk} are too large for a certain bus :

   (a) Calculate new load impedances according to $\overline{Z}_{LDk} = U_k^2/\overline{S}^*_{LDk}$ where U_k is the actual calculated voltage at bus k obtained in step 3,

   (b) Go back to step 2 and repeat the calculations.

   If ΔP_{LDk} and ΔQ_{LDk} are found to be acceptable for all power loads, the iteration process is finished.

A simple example will be given to clarify this method.

Example 6.4 *Assume a line operating with a voltage of* $\overline{U}_1 = 225\angle 0^\circ$ *kV in the sending end, i.e. bus 1, and with a load of* P_{LD} *= 80 MW and* Q_{LD} *= 60 MVAr in the receiving end, i.e. bus 2. The line has a length of 100 km with* x *= 0.4* Ω*/km,* r *= 0.04* Ω*/km and* $b_c = 3 \times 10^{-6}$ *S/km. Calculate the receiving end voltage.*

Solution

In Figure 6.19, the network modeled by impedance loads is given.

Assume $S_{base} = 100$ MVA and $U_{base} = 225$ kV which gives
$Z_{base} = U_{base}^2/S_{base} = 506.25\ \Omega$

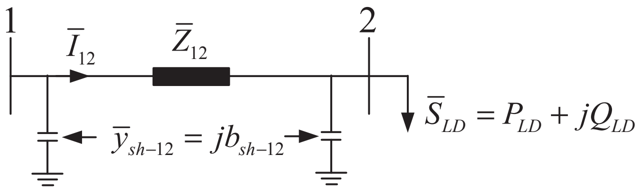


Figure 6.19. Impedance diagram of the system in Example 6.4.

This gives the following per-unit values of the line
$U_{1pu} = 225/U_{base} = 1.0$, $P_{LDpu} = P_{LD}/S_{base} = 0.8$, $Q_{LDpu} = Q_{LD}/S_{base} = 0.6$
$\overline{Z}_{12pu} = \mathcal{L}\,(r + j\,x)/Z_{base} = 100\,(0.04 + j\,0.4)/Z_{base} = 0.0079 + j\,0.0790$
Based on equation (3.9),
$b_{sh-12pu} = \mathcal{L}\,b_c\,Z_{base}/2 = 100\,(3 \times 10^{-6})\,Z_{base}/2 = 0.0759$
The iteration process can now be started :
1. $U_{2pu} = 1$, $\overline{Z}_{LDpu} = U_{2pu}^2/(P_{LDpu} - jQ_{LDpu}) = 0.8 + \text{j}0.6$
2. $\overline{I}_{12pu} = U_{1pu}/(\overline{Z}_{12pu} + \frac{1}{jb_{sh-12pu}} \parallel \overline{Z}_{LDpu}) = 0.7330 - \text{j}0.5415 \Rightarrow\ U_{2pu} = |U_{1pu} - \overline{I}_{12pu}\overline{Z}_{12pu}|$ $= 0.9529$
3. $\overline{S}_{LDpu} = U_{2pu}^2/\overline{Z}_{LDpu}^* = 0.7265 + \text{j}0.5448$
4. $\Delta P_{LD} = 0.0735$, $\Delta Q_{LD} = 0.0552$
1. $\overline{Z}_{LDpu} = 0.7265 + \text{j}0.5448$
2. $U_{2pu} = 0.9477$
4. $\Delta P_{LD} = 0.0087$, $\Delta Q_{LD} = 0.0066$
1. $\overline{Z}_{LDpu} = 0.7185 + \text{j}0.5389$
2. $U_{2pu} = 0.9471$
4. $\Delta P_{LD} = 0.0011$, $\Delta Q_{LD} = 0.00079$
1. $\overline{Z}_{LDpu} = 0.7176 + 0.5382\text{i}$
2. $U_{2pu} = 0.9470$
4. $\Delta P_{LD} = 0.0001$, $\Delta Q_{LD} = 0.0001$
This is found to be acceptable, which gives a voltage magnitude in the sending end of $U_2 = 0.9470 \cdot U_{base} = 213.08$ kV. This simple example can be solved exactly by using a non-linear expression which will be shown in Example 7.4.

Example 6.5 *Consider the system in Example 6.2, but let the short line model be used for the line (i.e. $b_c = 0$), and the load be considered as a constant power load.*

Calculate the voltage level at the industry.

Solution
1. From Example 6.2, we have the following:

$\overline{U}_{Thpu} = 1, \quad \overline{Z}_{Thpu} = j0.0137, \quad \overline{Z}_{tpu} = j0.004, \quad \overline{Z}_{21pu} = 0.0225 + j0.0075$

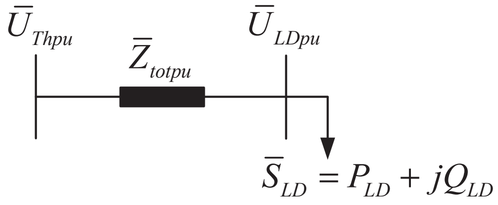


Figure 6.20. Network used in Example 6.5.

The total impedance between bus 1 and bus 4 is given by:
$\overline{Z}_{totpu} = \overline{Z}_{Thpu} + \overline{Z}_{tpu} + \overline{Z}_{21pu} = 0.0225 + j0.0252$

Calculate the per-unit values of the power demand of the industry as well as the corresponding impedance at nominal voltage :
$\overline{S}_{LDpu} = (P_{LD} + j[P_{LD}/\cos\phi] \cdot \sin\phi)/S_{base} = 0.8000 + j0.6000$
$\overline{Z}_{LDpu} = (U_n^2/\overline{S}^*_{LDpu})/U^2_{base10} = 0.8 + j0.6$

2. The Y-bus matrix of the network can be calculated as :

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{\overline{Z}_{totpu}} & -\frac{1}{\overline{Z}_{totpu}} \\ -\frac{1}{\overline{Z}_{totpu}} & \frac{1}{\overline{Z}_{totpu}} + \frac{1}{\overline{Z}_{LDpu}} \end{bmatrix} = \begin{bmatrix} 19.67 - j22.08 & -19.67 + j22.08 \\ -19.67 + j22.08 & 20.47 - j22.68 \end{bmatrix} \tag{6.63}$$

The Z-bus matrix is calculated as the inverse of the Y-bus matrix :

$$\mathbf{Z} = \mathbf{Y^{-1}} = \begin{bmatrix} 0.82 + j0.63 & 0.80 + j0.60 \\ 0.80 + j0.60 & 0.80 + j0.60 \end{bmatrix} \tag{6.64}$$

The voltage at the industry is now calculated according to equation (6.36) :

$$\overline{U}_{LDpu} = \mathbf{Z}(2,1) \cdot \overline{U}_{Thpu}/\mathbf{Z}(1,1) = 0.9679\angle - 0.3714° \tag{6.65}$$

3. The power delivered to the industry can be calculated as :

$$\overline{S}_{LDpu-b} = U^2_{LD}/\overline{Z}^*_{LDpu} = 0.7495 + j0.5621 \tag{6.66}$$

4. The difference between calculated and specified power can be calculated as :

$$\Delta P_{LD} = |Re(\overline{S}_{LDpu-b}) - Re(\overline{S}_{LDpu})| = 0.0505 \tag{6.67}$$
$$\Delta Q_{LD} = |Im(\overline{S}_{LDpu-b}) - Im(\overline{S}_{LDpu})| = 0.0379 \tag{6.68}$$

5. These deviations are too large and the calculations are therefore repeated and a new industry impedance is calculated by using the new voltage magnitude :

$$\overline{Z}_{LDpu} = (U^2_{LDpu}/\overline{S}^*_{LDpu}) = 0.7495 + j0.5621 \tag{6.69}$$

Repeat the calculations from step 2.

2, 3. $\Rightarrow \overline{S}_{LDpu-b} = 0.7965 + j0.5974$
4. $\Delta P_{LD} = 0.0035$, $\Delta Q_{LD} = 0.0026$

Unacceptable $\Rightarrow$
5. $\overline{Z}_{LDpu} = 0.7462 + j0.5597$
Continue from step 2.

2, 3. $\Rightarrow \overline{S}_{LDpu-b} = 0.7998 + j0.5998$
4. $\Delta P_{LD} = 0.00024$, $\Delta Q_{LD} = 0.00018$
Unacceptable $\Rightarrow$
5. $\overline{Z}_{LDpu} = 0.7460 + j0.5595$
Continue from step 2.

2, 3. $\Rightarrow \overline{S}_{LDpu-b} = 0.8000 + j0.6000$
4. $\Delta P_{LD} = 0.000016$, $\Delta Q_{LD} = 0.000012$
Acceptable $\Rightarrow$
$\overline{U}_{LD} = \overline{U}_{LDpu} \cdot U_{base10} = 9.6565\angle - 0.3974°$

Chapter 7

Power flow calculations

In chapters 5 and 6, it was assumed that the network had only one voltage source (or generator bus), and the loads were modelled as impedances. These assumptions resulted in using a linear set of equations which could be easily solved.

In this chapter, the loads are modelled as constant power loads, and the system has more than one generator bus (i.e. a multi-generator system).

First, the power flow in a transmission line will be derived, and then a more general power flow calculations (commonly known as load flow) will be presented.

7.1 Power flow in a line

Consider the the π-equivalent model of a line shown in Figure 7.1, where all variables expressed in per-unit.

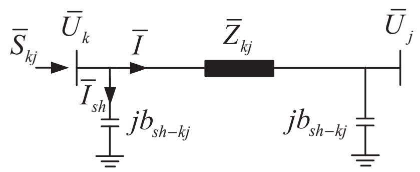


Figure 7.1. π-equivalent model of a line.

Let

$$\begin{aligned}
\overline{U}_k &= U_k e^{j\theta_k} \quad , \quad \overline{U}_j = U_j e^{j\theta_j} \\
\overline{Z}_{kj} &= R_{kj} + jX_{kj} \quad , \quad Z_{kj} = \sqrt{R_{kj}^2 + X_{kj}^2} \\
\theta_{kj} &= \theta_k - \theta_j
\end{aligned} \tag{7.1}$$

The power $\overline{S}_{kj}$ in the sending end k is given by

$$\begin{aligned}
\overline{S}_{kj} &= \overline{U}_k \left(\overline{I}_{sh}^* + \overline{I}^*\right) = \overline{U}_k \left(\left(j\, b_{sh-kj}\, \overline{U}_k\right)^* + \frac{\overline{U}_k^* - \overline{U}_j^*}{\overline{Z}_{kj}^*} \right) = \\
&= -j\, b_{sh-kj}\, U_k^2 + \frac{U_k^2}{R_{kj} - jX_{kj}} - \frac{U_k U_j}{R_{kj} - jX_{kj}} e^{j(\theta_k - \theta_j)} = \\
&= -j\, b_{sh-kj}\, U_k^2 + \frac{U_k^2}{Z_{kj}^2}(R_{kj} + jX_{kj}) - \frac{U_k U_j}{Z_{kj}^2}(R_{kj} + jX_{kj})\left(\cos\theta_{kj} + j\sin\theta_{kj}\right)
\end{aligned} \tag{7.2}$$

By dividing equation (7.2) into a real and an imaginary part, expressions for the active and reactive power can be obtained, respectively, as

$$\begin{aligned} P_{kj} &= \frac{R_{kj}}{Z_{kj}^2} U_k^2 + \frac{U_k U_j}{Z_{kj}^2} (X_{kj} \sin\theta_{kj} - R_{kj} \cos\theta_{kj}) \\ &= \frac{R_{kj}}{Z_{kj}^2} U_k^2 + \frac{U_k U_j}{Z_{kj}} \sin\left(\theta_{kj} - \arctan\left(\frac{R_{kj}}{X_{kj}}\right)\right) \end{aligned} \tag{7.3}$$

$$\begin{aligned} Q_{kj} &= -b_{sh-kj} U_k^2 + \frac{X_{kj}}{Z_{kj}^2} U_k^2 - \frac{U_k U_j}{Z_{kj}^2} (R \sin\theta_{kj} + X_{kj} \cos\theta_{kj}) \\ &= \left(-b_{sh-kj} + \frac{X_{kj}}{Z_{kj}^2}\right) U_k^2 - \frac{U_k U_j}{Z_{kj}} \cos\left(\theta_{kj} - \arctan\left(\frac{R_{kj}}{X_{kj}}\right)\right) \end{aligned} \tag{7.4}$$

From equations (7.3) and (7.4), it can be concluded that if the phasor voltages (i.e. the voltage magnitude and phase angle) at both ends of the line are known, the power flow can be uniquely determined. This implies that if the phasor voltages of all buses in a system are known, the power flows in the whole system are known, i.e the phasor voltages define the system state.

Example 7.1 *Assume a line where the voltage in the sending end is* $\overline{U}_1 = 225\angle 0°$ *kV and in the receiving end* $\overline{U}_2 = 213.08\angle -3.572°$ *kV. The line has a length of 100 km and has* $x = 0.4\ \Omega/km$, $r = 0.04\ \Omega/km$ *and* $b_c = 3 \times 10^{-6}$ *S/km. Calculate the amount of power transmitted from bus 1 to bus 2.*

Solution

Assume $S_{base} = 100$ MVA and $U_{base} = 225$ kV, this gives that
$Z_{base} = U_{base}^2/S_{base} = 506.25\ \Omega$
The per-unit values for the line are
$U_1 = 225/U_{base} = 1.0$ pu, $U_2 = 213.08/U_{base} = 0.9470$ pu, $\theta_{12} =$ 0-(-3.572) $= 3.572°$
$R_{12} = 0.04 \cdot 100/Z_{base} = 0.0079$ pu, $X_{12} = 0.4 \cdot 100/Z_{base} = 0.0790$ pu,
$b_{sh-12} = 3 \times 10^{-6} \cdot 100 \cdot Z_{base}/2 = 0.0759$ pu, $Z_{12} = \sqrt{R_{12}^2 + X_{12}^2} = 0.0794$ pu
The power flow in per-unit can be calculated by using equation (7.3) and (7.4) :

$$\begin{aligned} P_{12} &= \frac{0.0079}{0.0794^2} 1.0^2 + \frac{1.0 \cdot 0.9470}{0.0794} \sin\left(3.572° - \arctan\left(\frac{0.0079}{0.0790}\right)\right) = \\ &= 0.8081 \text{ pu} \\ Q_{12} &= \left(-0.0759 + \frac{0.0790}{0.0794^2}\right) * 1.0^2 - \frac{1.0 \cdot 0.9470}{0.0794} \cos\left(3.572° - \arctan\left(\frac{0.0079}{0.0790}\right)\right) = \\ &= 0.5373 \text{ pu} \end{aligned}$$

expressed in nominal values

$$\begin{aligned} P_{12} &= 0.8081 \cdot S_{base} = 80.81 \text{ MW} \\ Q_{12} &= 0.5373 \cdot S_{base} = 53.73 \text{ MVAr} \end{aligned}$$

For this simple system, the calculations can be performed without using the per-unit system. By using equation (5.7), equation (7.3) can be rewritten as

$$\begin{aligned} P_{kj}(\text{MW}) &= P_{kj}(\text{pu})S_{base} = \frac{U_{base}^2}{Z_{base}}\left\{\frac{R_{kj}}{Z_{kj}^2}U_k^2 + \frac{U_k\,U_j}{Z_{kj}}\sin\left(\theta_{kj} - \arctan\left(\frac{R_{kj}}{X_{kj}}\right)\right)\right\} = \\ &= \frac{R_{kj}(\Omega)}{Z_{kj}^2(\Omega)}U_k^2(\text{kV}) + \frac{U_k(\text{kV})\cdot U_j(\text{kV})}{Z_{kj}(\Omega)}\sin\left(\theta_{kj} - \arctan\left(\frac{R_{kj}}{X_{kj}}\right)\right) \end{aligned}$$

i.e. this equation is the same independent on if the values are given as nominal or per-unit values. Note that $\arctan(R_{kj}/X_{kj}) = \arctan(R_{kj}(\Omega)/X_{kj}(\Omega))$.

For a high voltage overhead line (U > 70 kV), the line reactance is normally considerably higher than the resistance of the line, i.e. $R_{kj} \ll X_{kj}$ in equation (7.3). An approximate form of that equation is (i.e. $R_{kj} \approx 0$)

$$P_{kj} \approx \frac{U_k U_j}{X_{kj}}\sin\theta_{kj} \tag{7.5}$$

i.e. the sign of θ_{kj} determines the direction of the active power flow on the line. Normally, the active power will flow towards the bus with the lowest voltage angle. This holds also for lines having a pronounced resistivity.

Assume that the voltages $\overline{U}_k$ and $\overline{U}_j$ are in phase and that the reactance of the line is dominating the line resistance (i.e. $R \approx 0$). This implies that the active power flow is very small. Equation (7.4) can be rewritten as

$$Q_{kj} = -b_{sh-kj}\,U_k^2 + \frac{U_k(U_k - U_j)}{X_{kj}} \tag{7.6}$$

Equation (7.6) indicates that this type of line gives a reactive power flow towards the bus with the lowest voltage magnitude. The equation shows that if the difference in voltage magnitude between the ends of the line is small, the line will generate reactive power. This since the reactive power generated by the shunt admittances in that case dominates the reactive power consumed by the series reactance. The "rule of thumb" that reactive power flows towards the bus with lowest voltage is more vague than the rule that active power flows towards the bus with lowest angle. The fact that overhead lines and especially cables, generates reactive power when the active power flow is low, is important to be aware of.

Example 7.2 *Using the approximate expressions* (7.5) *and* (7.6)*, respectively, calculate the active and reactive power flow in the line in Example 7.1.*

Solution

$$\begin{aligned} P_{12} &\approx \frac{1.0\cdot 0.9470}{0.0790}\sin 3.572^\circ = 0.7468 \text{ pu} \Rightarrow 74.68 \text{ MW} \\ Q_{12} &\approx -0.0759\cdot 1.0^2 + \frac{1.0(1.0 - 0.9470)}{0.0790} = 0.5948 \text{ pu} \Rightarrow 59.48 \text{ MVAr} \end{aligned}$$

The answers are of right dimension and have correct direction of the power flow but the active power flow is about 8 % too low and the reactive power flow is 11 % too large.

7.1.1 Line losses

The active power losses on a three-phase line are dependent on the line resistance and the actual line current. By using physical units (i.e. not in per-unit), the losses can be calculated as

$$P_f = 3R_{kj}I^2 \tag{7.7}$$

The squared current dependence in equation (7.7) can be written as

$$I^2 = Ie^{j\gamma}\,Ie^{-j\gamma} = \overline{I}\,\overline{I}^* = \frac{\overline{S}^*}{\sqrt{3}\overline{U}^*}\frac{\overline{S}}{\sqrt{3}\overline{U}} = \frac{S^2}{3U^2} = \frac{P^2+Q^2}{3U^2} \tag{7.8}$$

The active power losses for the line given in Figure 7.1 can be calculated as

$$P_f = R_{kj}\frac{P_{kj}^2 + (Q_{kj} + b_{sh-kj}\,U_k^2)^2}{U_k^2} \tag{7.9}$$

where, $b_{sh-kj}\,U_k^2$ is the reactive power generated by the shunt capacitance at bus k.

The expression given by (7.9) is valid both for nominal and for per-unit values. This equation shows that a doubling of transmitted active power will increase the active power losses by a factor of four. If the voltage is doubled, the active power losses will decrease with a factor of four.

Assume that the active power injections at both ends of the line are known, i.e. both P_{kj} and P_{jk} have been calculated using equation (7.3). The active power losses can then be calculated as

$$P_f = P_{kj} + P_{jk} \tag{7.10}$$

The reactive power losses can be obtained in the corresponding manner

$$Q_f = 3X_{kj}I^2 = X_{kj}\frac{P_{kj}^2 + (Q_{kj} + b_{sh-kj}\,U_k^2)^2}{U_k^2} \tag{7.11}$$

Equations (7.8) and (7.9) shows that the losses are proportional to S^2 and that the losses will increase if reactive power is transmitted over the line. A natural solution to that is to generate the reactive power as close to the consumer as possible. Of course, active power is also generated as close to the consumer as possible, but the generation costs are of great importance.

Example 7.3 *Use the same line as in Example 7.1 and calculate the active power losses.*

Solution

The losses on the line can be calculated by using equation (7.9) and the conditions that apply at the sending end

$$\begin{aligned} P_f(\mathrm{MW}) &= R_{12}\frac{P_{12}^2 + (Q_{12} + b_{sh-12}U_1^2)^2}{U_1^2}S_{base} = \\ &= 0.0079\frac{0.8081^2 + (0.5373 + 0.0759\cdot 1.0^2)^2}{1.0^2}100 = 0.81\ \mathrm{MW} \end{aligned}$$

The losses can also be calculated by using the receiving end conditions

$$\begin{aligned} P_f(\text{MW}) &= R_{12}\frac{P_{21}^2 + (Q_{21} + b_{sh-12}U_2^2)^2}{U_2^2}S_{base} = \\ &= 0.0079\frac{(-0.80)^2 + (-0.60 + 0.0759 \cdot 0.9470^2)^2}{0.9470^2}100 = 0.81 \text{ MW} \end{aligned}$$

or by using equation (7.10)

$$P_f(\text{MW}) = [P_{12} + P_{21}]S_{base} = [0.8081 + (-0.80)]100 = 0.81 \text{ MW}$$

7.1.2 Shunt capacitors and shunt reactors

As mentioned earlier in subsection 7.1.1, transmission of reactive power will increase the line losses. An often used solution is to generate reactive power as close to the load as possible. This is done by switching in *shunt capacitors*. Figure 7.2 shows a Y-connected shunt capacitor. Figure 7.2 also shows the single-phase equivalent which can be used at

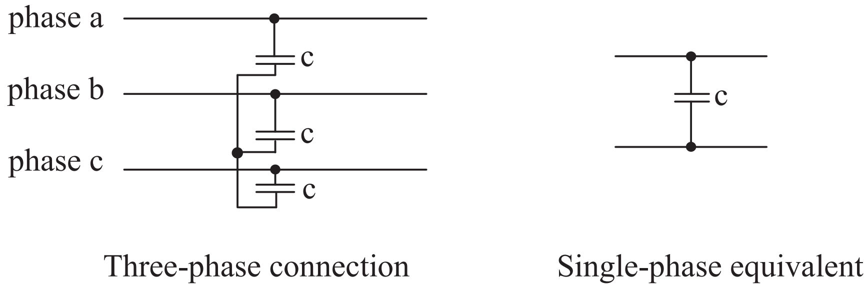


Figure 7.2. Y-connected shunt capacitors.

symmetrical conditions. A shunt capacitor generates reactive power proportional to the bus voltage squared U^2. In the per-unit system, we have

$$Q_{sh} = B_{sh}U^2 = 2\pi f c U^2 \tag{7.12}$$

An injection of reactive power into a certain bus will increase the bus voltage, see Example 7.6. The insertion of shunt capacitors in the network is also called *phase compensation*. This because the phase displacement *between voltage and current* is reduced when the reactive power transmission through the line is reduced.

As mentioned earlier, lines that are lightly loaded generates reactive power. The amount of reactive power generated is proportional to the length of the line. In such situations, the reactive power generation will be too large and it is necessary to consume the reactive power in order to avoid overvoltages. One possible countermeasure is to connect *shunt reactors*. They are connected and modeled in the same way as the shunt capacitors with the difference that the reactors *consume* reactive power.

7.1.3 Series capacitors

By studying equation (7.5), an approximate expression of the maximum amount of power that can be transmitted through a line, at a certain voltage level, can be written as

$$P_{kj-max} \approx \max_{\theta_{kj}} \frac{U_k U_j}{X_{kj}} \sin \theta_{kj} = \frac{U_k U_j}{X_{kj}} \tag{7.13}$$

i.e. the larger the reactance of the line is, the less amount of power can be transmitted. One possibility to increase the maximum loadability of a line is to compensate for the series reactance of the line by using series capacitors. In Figure 7.3, the way of connecting series capacitors is shown as well as the single-phase equivalent of a series compensated line. The

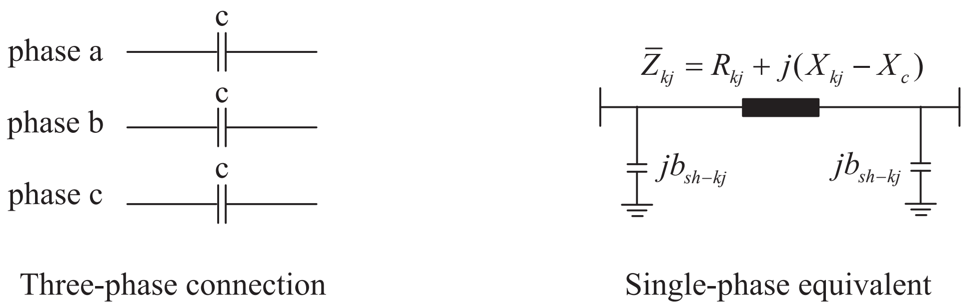


Figure 7.3. Series capacitors

expression for the maximum loadability of a series compensated line is

$$P_{kj-max} \approx \frac{U_k U_j}{X_{kj} - X_c} \tag{7.14}$$

It is obvious that the series compensation increases the loadability of the line.

The use of series capacitors will also reduce the voltage drop along the line, see Example 7.7.

7.2 Non-linear power flow equations

The technique of determining all bus voltages in a network is usually called *load flow*. When knowing the voltage magnitude and voltage angle at all buses, the system state is completely determined and all system properties of interest can be calculated, e.g. line loadings and line losses.

In a power system, power can be generated and consumed at many different locations. Consider now a balanced power system with N buses. Figure 7.4 schematically shows connection of the system components to bus k.

The generator generates the current $\overline{I}_{Gk}$ (in pu), the load at the bus draws the current $\overline{I}_{LDk}$ (in pu), and $\overline{I}_{kj}$ (in pu) is the currents from bus k to the neighboring buses. According to

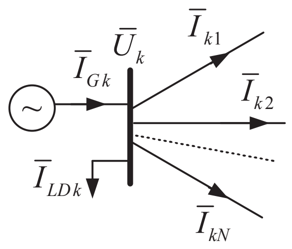


Figure 7.4. Notation of bus k in a network.

Kirchoff's current law, the sum of all currents injected into bus k must be zero, i.e.

$$\overline{I}_{Gk} - \overline{I}_{LDk} = \sum_{j=1}^{N} \overline{I}_{kj} \tag{7.15}$$

By taking the conjugate of equation (7.15) and multiply the equation with the bus voltage (in pu), the following holds

$$\overline{U}_k \overline{I}^*_{Gk} - \overline{U}_k \overline{I}^*_{LDk} = \sum_{j=1}^{N} \overline{U}_k \overline{I}^*_{kj} \tag{7.16}$$

This can be rewritten as an expression for complex power in the per-unit system as

$$\overline{S}_{Gk} - \overline{S}_{LDk} = \sum_{j=1}^{N} \overline{S}_{kj} \tag{7.17}$$

where

$\overline{S}_{Gk} = P_{Gk} + jQ_{Gk}$ is the generated complex power at bus k,

$\overline{S}_{LDk} = P_{LDk} + jQ_{LDk}$ is the consumed complex power at bus k,

$\overline{S}_{kj} = P_{kj} + jQ_{kj}$ is the complex power flow from bus k to bus j.

The power balance at the bus according to equation (7.17) must hold both for the active and for the reactive part of the expression. By using P_{GDk} and Q_{GDk} as notation for the net generation of active and reactive power at bus k, respectively, the following expression holds

$$P_{GDk} = P_{Gk} - P_{LDk} = \sum_{j=1}^{N} P_{kj} \tag{7.18}$$

$$Q_{GDk} = Q_{Gk} - Q_{LDk} = \sum_{j=1}^{N} Q_{kj} \tag{7.19}$$

i.e. for any bus k in the system, the power balance must hold for both active and reactive power. Note that in equations (7.18)-(7.19), P_{LDk} and Q_{LDk} are assumed constant.

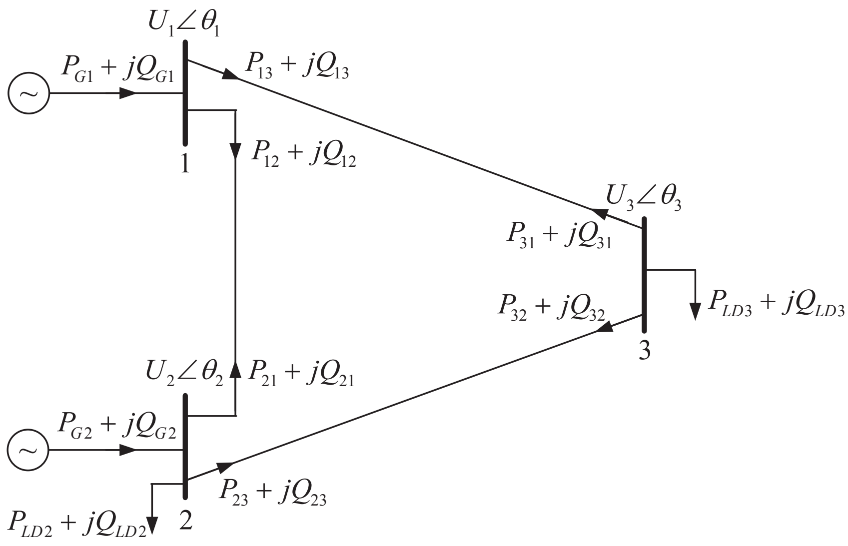


Figure 7.5. Single-line diagram of a balanced three-bus power system.

Figure 7.5 shows a balanced power system with $N = 3$. Based on equations (7.18) and (7.19), the following system of equations can be obtained.

$$\left\{\begin{array}{rcl} P_{G1} & = & P_{12} + P_{13} \\ Q_{G1} & = & Q_{12} + Q_{13} \\ P_{G2} - P_{LD2} & = & P_{21} + P_{23} \\ Q_{G2} - Q_{LD2} & = & Q_{21} + Q_{23} \\ -P_{LD3} & = & P_{31} + P_{32} \\ -Q_{LD3} & = & Q_{31} + Q_{32} \end{array}\right. \tag{7.20}$$

At each bus in Figure 7.5, four variables are of interest: net generation of active power P_{GDk}, net generation of reactive power Q_{GDk}, voltage magnitude U_k and voltage phase angle θ_k. This gives that the total number of variables for the system are $3 \cdot 4 = 12$. The voltage phase angles must be given as an angle in relation to a *reference angle*. This since the phase angles are only relative to one another and not absolute. This reduces the number of system variables to $12 - 1 = 11$. However, there are only six equations in the system of equations (7.20), this gives that five quantities must be known to be able to solve for the remaining six variables. Depending on what quantities that are known at a certain bus, the buses are mainly modeled in three different types.

PQ-bus, Load bus : For this bus, the net generated power P_{GDk} and Q_{GDk} are assumed to be known. The name PQ-bus is based on that assumption. On the other hand, the voltage magnitude U_k and the voltage phase angle θ_k are unknown. A PQ-bus is most often a bus with a pure load demand, as bus 3 in Figure 7.5. It represents a system bus where the power consumption can be considered to be independent of the voltage magnitude. This model is suitable for a load bus located on the low voltage side of a regulating transformer. The regulating transformer keeps the load voltage constant independent of the voltage fluctuations on the high voltage side of the transformer. Note that a PQ-bus can be a bus without generation as well as load, i.e. $P_{GDk} = Q_{GDk} = 0$. This holds e.g. at a bus where a line is connected to a transformer.

PU-bus, Generator bus : In a PU-bus, the net active power generation P_{GDk} as well as the voltage magnitude U_k are assumed to be known. This gives that the net reactive power generation Q_{GDk} and the voltage angle θ_k are unknown. In a PU-bus some sort of voltage regulating device must be connected since the voltage magnitude is independent of the net reactive power generation. For example, in a synchronous machine, the terminal voltage can be regulated by changing the magnetizing current. In a system, voltage can be regulated by using controllable components as controllable shunt capacitors and controllable shunt reactors. A standard component is called SVC, Static VAr Compensator. This component change the reactive power flow in order to regulate the bus voltage. Assume that bus 2 in Figure 7.5 is modeled as a PU-bus. This gives that the active power generation of the generator as well as the active power consumption of the load are known. Also the reactive power consumption of the load is known. The bus voltage is constant due to the magnetization system of the generator. The generator may generate or consume reactive power in such a way that the relation in equation (7.19) holds.

$U\theta$-bus, Slack bus : At a slack bus (only one bus in each system), the voltage magnitude and the voltage phase angle are known and fixed. The voltage phase angle is chosen as a the reference phase angle in the system. Normally, the phase angle θ_k is set to zero. Unknown quantities are the net generation of both active and reactive power. At this bus, (as for the PU-bus) a voltage regulating component must be present. Since the active power is allowed to vary, a generator or an active power in-feed into the system is assumed to exist at this bus. Since this bus also is the *only* bus where the active power is allowed to vary, the slack bus will take care of the system losses since they are unknown. If the loads have been modeled in the load flow as constant power loads and a line is tripped, the only bus which will change the active power generation is the slack bus. If bus 1 is chosen as slack bus in Figure 7.5, both P_{G1} and Q_{G1} are unknown but the voltage U_1 is given as well as the reference angle $\theta_1 = 0$.

Assume that M of the system N buses are PU-buses. A summary of the different bus types is given in Table 7.1. As given in equations (7.3)–(7.4), the active and reactive power flow

Bus model	Number	Known quantities	Unknown quantities
$U\theta$-bus, Slack bus	1	U, θ	P_{GD}, Q_{GD}
PU-bus, Generator bus	M	P_{GD}, U	Q_{GD}, θ
PQ-bus, Load bus	N-M-1	P_{GD}, Q_{GD}	U, θ

Table 7.1. Bus types for load flow calculations

through a line can be expressed as a function of the voltage magnitude and voltage phase angle at both ends of the line. Assume that the power system in Figure 7.5 is modeled in such a way that bus 1 is a slack bus, bus 2 is a PU-bus and bus 3 is a PQ-bus. By using this bus type modeling, the system of equations (7.20) can be written as

$$\begin{cases} P_{GD1}(\text{unknown}) &= P_{12}(U_1,\theta_1,U_2,\theta_2) + P_{13}(U_1,\theta_1,U_3,\theta_3) \\ Q_{G1}(\text{unknown}) &= Q_{12}(U_1,\theta_1,U_2,\theta_2) + Q_{13}(U_1,\theta_1,U_3,\theta_3) \\ P_{GD2} &= P_{21}(U_1,\theta_1,U_2,\theta_2) + P_{23}(U_2,\theta_2,U_3,\theta_3) \\ Q_{GD2}(\text{unknown}) &= Q_{21}(U_1,\theta_1,U_2,\theta_2) + Q_{23}(U_2,\theta_2,U_3,\theta_3) \\ P_{GD3} &= P_{31}(U_1,\theta_1,U_3,\theta_3) + P_{32}(U_2,\theta_2,U_3,\theta_3) \\ Q_{GD3} &= Q_{31}(U_1,\theta_1,U_3,\theta_3) + Q_{32}(U_2,\theta_2,U_3,\theta_3) \end{cases} \tag{7.21}$$

where also θ_2, U_3 and θ_3 are unknown quantities whereas the others are known. As given in equation (7.21), unknown power quantities appear only on the left hand side for buses modeled as slack and PU-bus. These quantities can be easily calculated when voltage magnitudes and angles are known. These equations are not contributing to the system of equations since they only give one extra equation, and one extra variable which easily can be calculated. The system of equations in (7.21) can therefore be simplified to a system of equations containing unknown U and θ as

$$\begin{cases} P_{GD2} &= P_{21}(U_1,\theta_1,U_2,\theta_2) + P_{23}(U_2,\theta_2,U_3,\theta_3) \\ P_{GD3} &= P_{31}(U_1,\theta_1,U_3,\theta_3) + P_{32}(U_2,\theta_2,U_3,\theta_3) \\ Q_{GD3} &= Q_{31}(U_1,\theta_1,U_3,\theta_3) + Q_{32}(U_2,\theta_2,U_3,\theta_3) \end{cases} \tag{7.22}$$

The system of equations given by (7.22) is non-linear since the expressions for power flow through a line (equation (7.3)–(7.4)) include squared voltages as well as trigonometric expressions. This system of equations can e.g. be solved by using the Newton-Raphson method.

The system of equations given by (7.22) can be generalized to a system containing N buses, of which M have a voltage regulating device in operation. A summary of this system is given in Table 7.2. As indicated in Table 7.2, the system of equations contains as many unknown

Bus model	Number	Balance equations		Unknown quantities	
		$P_{GDk} = \sum P_{kj}$	$Q_{GDk} = \sum Q_{kj}$	U_k	θ_k
Slack bus	1	0	0	0	0
PU-bus	M	M	0	0	M
PQ-bus	N-M-1	N-M-1	N-M-1	N-M-1	N-M-1
Total	N	2N-M-2		2N-M-2	

Table 7.2. Summary of equations and unknown quantities at load flow calculations

quantities as the number of equations, and by that, the system is solvable.

7.3 Power flow calculations of a simple two-bus system

As shown in section 7.2, constant power loads give a non-linear system of equations, and power flow calculations for large power system requires soft-ware tools such as MATLAB which will be used in this course. To understand the concept of power flow calculations, in this section a simple two-bus system is studied. Since for power flow calculations, a bus bus must be a slack bus, there are therefore two possible bus-type combinations, namely, slack bus + PU-bus and slack bus + PQ-bus which can be analytically handled.

Consider the two-bus power system shown in Figure 7.6. The data given in Example 7.1 is used for this system. Let bus 1 be a slack bus with $\overline{U}_1 = 225\angle 0°$ kV. Let also $P_{LD2} = 80$ MW.

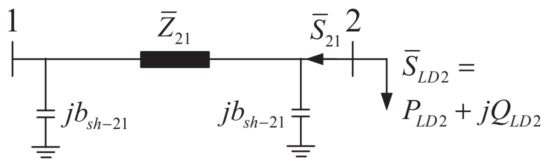


Figure 7.6. Single-line diagram of a balanced two-bus power system.

7.3.1 Slack bus + PU-bus

This combination is of interest when the voltage magnitude is known at both buses and the net active power (i.e. P_{GD}) is known at one of the buses. This implies that the only unknown quantity is the voltage phase angle at the PU-bus, i.e. the bus having a known net active power P_{GD}.

Example 7.4 *Let bus 2 be a PU-bus with $U_2 = 213.08$ kV. Calculate the voltage phase angle at bus 2 (the same as Example 6.4). Find also Q_{21} in MVAr.*

Solution

From equation (7.3) we have

$$\begin{aligned}
P_{21} &= \frac{R_{21}}{Z_{21}^2} U_2^2 + \frac{U_2 U_1}{Z_{21}} \sin\left(\theta_{21} - \arctan\left(\frac{R_{21}}{X_{21}}\right)\right) \quad \Rightarrow \\
\theta_2 &= \theta_1 + \arctan\left(\frac{R_{21}}{X_{21}}\right) + \arcsin\left(\frac{Z_{21}}{U_2 U_1}\left(P_{21} - \frac{R_{21}}{Z_{21}^2} U_2^2\right)\right)
\end{aligned}$$

where,

$$P_{21} = P_{GD2} = (0 - P_{LD2})/S_{base} = -0.8 \quad \text{pu}, \quad \arctan\frac{R_{21}}{X_{21}} = \arctan\frac{0.0079}{0.0790} = 5.71°$$

Thus,

$$\theta_2 = 0 + 5.71° + \arcsin\left(\frac{0.0794}{0.9470 \cdot 1.0}\left(-0.8 - \frac{0.0079}{0.0794^2} 0.9470^2\right)\right) = -3.5724°$$

From equation (7.4), we have

$$\begin{aligned}
Q_{21} &= \left[\left(-b_{sh-21} + \frac{X_{21}}{Z_{21}^2}\right) U_2^2 - \frac{U_2 U_1}{Z_{21}} \cos\left(\theta_{21} - \arctan\left(\frac{R_{21}}{X_{21}}\right)\right)\right] S_{base} \\
&= \left[\left(-0.0759 + \frac{0.0790}{0.0794^2}\right) 0.9470^2 - \frac{0.9470\,1.0}{0.0794} \cos(-3.5724° - 5.71°)\right] 100 \\
&= -59.9793 \quad \text{MVAr, from MATLAB}
\end{aligned}$$

7.3.2 Slack bus + PQ-bus

This combination is of interest when the voltage magnitude is known only at one of the buses and the net active and reactive power generation are known at the other bus.

Example 7.5 *Now let bus 2 be a PQ-bus, and $Q_{LD2} = -Q_{21}$ where Q_{21} has been obtained in Example 7.4. Calculate the voltage magnitude and phase angle at bus 2.*

Solution
Based on equations (7.3) and (7.4), by eliminating θ_{21}, the voltage magnitude U_2 can be analytically found, and it is given by

$$U_2^2 = -\frac{a_4}{2a_3} \underset{(-)}{+} \sqrt{\left(\frac{a_4}{2a_3}\right)^2 - \frac{1}{a_3}\left(a_1^2 + a_2^2\right)} \tag{7.23}$$

where

$$\begin{aligned} a_1 &= -R_{21}\, P_{21} - X_{21}\, Q_{21} \\ a_2 &= -X_{21}\, P_{21} + R_{21}\, Q_{21} \\ a_3 &= (1 - X_{21}\, b_{sh-21})^2 + R_{21}^2\, b_{sh-21}^2 \\ a_4 &= 2 \cdot a_1(1 - X_{21}\, b_{sh-21}) - U_1^2 + 2a_2 R_{21}\, b_{sh-21} \end{aligned}$$

The voltage U_2 can now be calculated as

$$U_2 = \underset{(-)}{+} \sqrt{U_2^2} \tag{7.24}$$

In our case,

$$\begin{aligned} a_1 &= -0.0079(-0.8) - 0.0790(-0.5998) = 0.0537 \\ a_2 &= -0.0790(-0.8) + 0.0079(-0.6)) = 0.0585 \\ a_3 &= (1 - 0.0790 \cdot 0.0759)^2 + 0.0079^2 \cdot 0.0759^2 = 0.9880 \\ a_4 &= 2 \cdot 0.0537(1 - 0.0790 \cdot 0.0759) - 1.0^2 + \\ &+ 2 \cdot 0.0585 \cdot 0.0079 \cdot 0.0759 = -0.8931 \\ &\Rightarrow \\ U_2^2 &= 0.4520 \underset{(-)}{+} 0.4449 = 0.8968 \\ &\Rightarrow \\ U_2 &= \underset{(-)}{+} \sqrt{0.8968} = 0.9470 \\ &\Rightarrow \\ U_2\ (\text{kV}) &= 0.9470 \cdot U_{base} = 213.08\ \text{kV} \end{aligned}$$

The voltage phase angle can now be calculated in the same way as performed in Example 7.4, which results in the the same answer.

Example 7.6 *Use the data given in Example 7.5 with* P_{D2} *= 80 MW and* $Q_{D2} \approx$ *60 MVAr. Use these load levels as a base case and calculate the voltage* U_2 *when the active and reactive load demand are varying between 0–100 MW and 0–100 MVAr, respectively.*

Solution

By using equations (7.23) and (7.24), the voltage can be calculated. The result is shown in Figure 7.7. The base case, i.e. P_{D2} = 80 MW and Q_{D2} = 60 MVAr, is marked by circles on

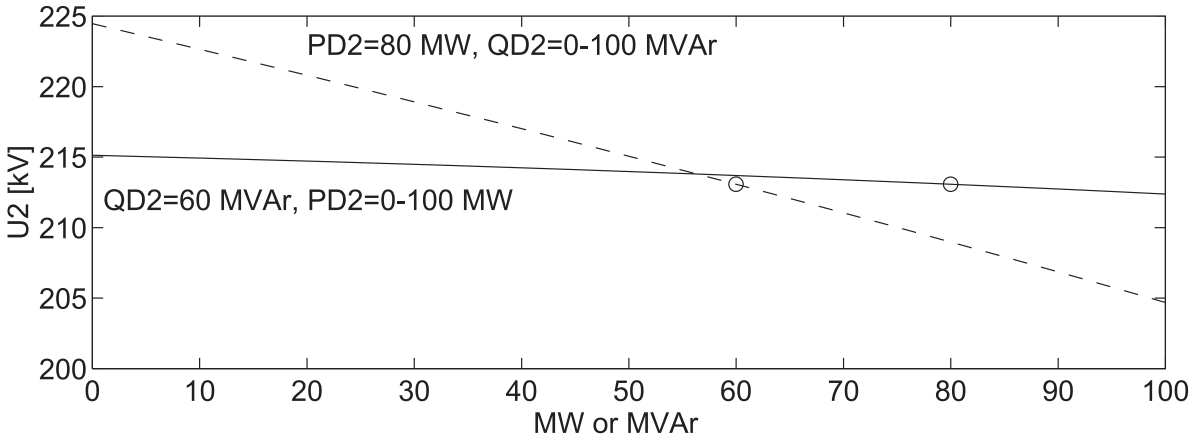


Figure 7.7. The voltage U_2 as a function of P_{D2} and Q_{D2}.

both curves. As shown in the figure, the voltage drops at bus 2 as the load demand increases. The voltage at bus 2 is much more sensitive to a change in reactive load demand compared to a change in active demand. If a shunt capacitor generating 10 MVAr is connected at bus 2 when having a reactive load demand of 60 MVAr, the net demand of reactive power will decrease to 50 MVAr and the bus voltage will increase by two kV, from 213 kV to 215 kV. As discussed earlier in subsection 7.1.1, a reduced reactive power load demand will also reduce the losses on the line.

Example 7.7 *Use the base case in Example 7.6, i.e.* P_{D2} *= 80 MW and* Q_{D2} *= 60 MVAr. Calculate the voltage* U_2 *when the series compensation of the line is varied in the interval 0–100 %.*

Solution

A series compensation of 0–100 % means that 0–100 % of the line reactance is compensated by series capacitors. 0 % means no series compensation at all and 100 % means that $X_c = X_{21}$. The voltage can be calculated by using equations (7.23) and (7.24). The result is shown in Figure 7.8. As shown in Figure 7.8, the voltage at bus 2 increases as the degree of series compensation increases. If the degree of compensation is 40 %, the voltage at bus 2 is increased by 4.5 kV (= 2 %) from 213.1 kV to 217.6 kV.

When having short lines or when only interested in approximate calculations, the shunt capacitance of a line can be neglected. In these conditions, b_{sh-21} in equation (7.23) is

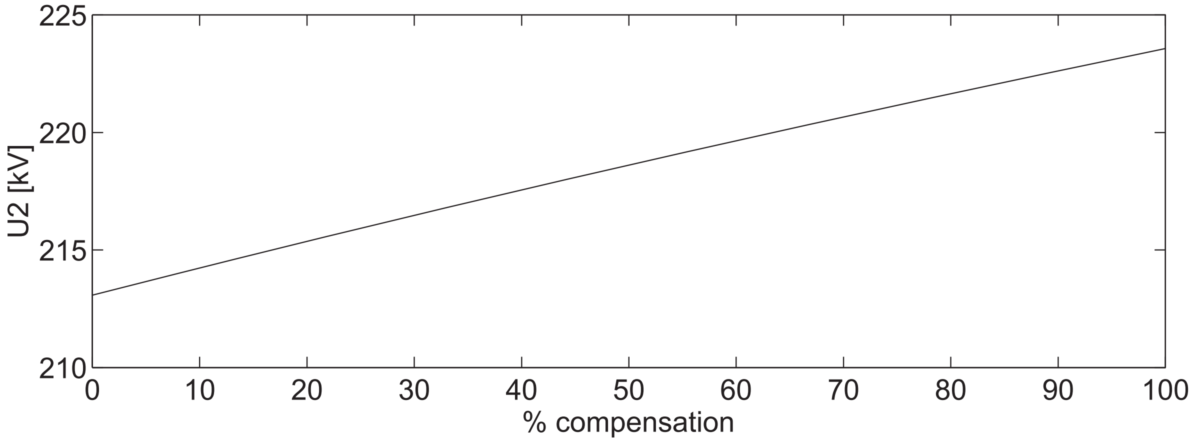


Figure 7.8. The voltage U_2 as a function of degree of compensation.

neglected, and the equation will be rewritten as

$$U_2 = \sqrt{\frac{U_1^2 - 2a_1}{2} \underset{(-)}{+} \sqrt{\left(\frac{U_1^2 - 2a_1}{2}\right)^2 - (a_1^2 + a_2^2)}} \tag{7.25}$$

where

$$\begin{aligned} a_1 &= -R_{21}\,P_{21} - X_{21}\,Q_{21} \\ a_2 &= -X_{21}\,P_{21} + R_{21}\,Q_{21} \end{aligned}$$

Example 7.8 *Use the data given in Example 7.5. Calculate the magnitude of the voltage by using the approximate expression given by equation* (7.25)*.*

Solution

Equation (7.25) gives that

$$\begin{aligned} a_1 &= -0.0790(-0.8) + 0.0079(-0.6) = 0.0537 \\ a_2 &= 0.0079(-0.6) - 0.0790(-0.8) = 0.0585 \\ &\Rightarrow \\ U_2 &= 0.9410 \\ &\Rightarrow \\ U_2(\text{kV}) &= 0.9410 \cdot U_{base} = 211.72 \text{ kV} \end{aligned}$$

i.e. the voltage becomes 0.6 % too low compared to the more accurate result.

Another approximation often used, is to neglect a_2 in equation (7.25). That equation can then be rewritten as

$$U_2 \approx \frac{U_1}{2} + \sqrt{\frac{U_1^2}{4} + R_{21}\,P_{21} + X_{21}\,Q_{21}} \tag{7.26}$$

Example 7.9 *Use the same line as in Example 7.5. Calculate the voltage by using the approximate expression given by equation* (7.26).

Solution

Equation (7.26) gives that

$$\begin{aligned} U_2 &= 0.9430 \\ &\Rightarrow \\ U_2(kV) &= 0.9439 \cdot U_{base} = 212.18 \text{ kV} \end{aligned}$$

i.e. the calculated voltage is 0.4 % too low. As indicated in this example, equation (7.26) gives a good approximation of the voltage drop on the line. In the equation, it is also clearly given that the active and reactive load demand have influence on the voltage drop. The reason why the voltage drop is more sensitive to a change in reactive power compared to a change in active power, is that the line reactance dominates the line resistance.

7.4 Newton-Raphson method

7.4.1 Theory

The Newton-Raphson method may be applied to solve for $x_1, x_2, \cdots, x_n$ of the following non-linear equations,

$$\begin{aligned} g_1(x_1, x_2, \cdots, x_n) &= f_1(x_1, x_2, \cdots, x_n) - b_1 = 0 \\ g_2(x_1, x_2, \cdots, x_n) &= f_2(x_1, x_2, \cdots, x_n) - b_2 = 0 \\ &\vdots \\ g_n(x_1, x_2, \cdots, x_n) &= f_n(x_1, x_2, \cdots, x_n) - b_n = 0 \end{aligned} \tag{7.27}$$

or in the vector form

$$g(x) = f(x) - b = 0 \tag{7.28}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad , \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix} \quad , \quad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad , \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

x is an $n \times 1$ vector which contains variables, b is an $n \times 1$ vector which contains constants, and $f(x)$ is an $n \times 1$ vector-valued function.

Taylor's series expansion of (7.28) is the basis for the Newton-Raphson method of solving (7.28) in an iterative manner. From an initial estimate (or guess) $x^{(0)}$, a sequence of gradually better estimates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $\cdots$ will be made that hopefully will converge to the solution x^*.

Let x^* be the solution of (7.28), i.e. $g(x^*) = 0$, and $x^{(i)}$ be an estimate of x^*. Let also $\Delta x^{(i)} = x^* - x^{(i)}$. Equation (7.28) can now be written as

$$g(x^*) = g(x^{(i)} + \Delta x^{(i)}) = 0 \tag{7.29}$$

Taylor's series expansion of (7.29) gives

$$g(x^{(i)} + \Delta x^{(i)}) = g(x^{(i)}) + JAC^{(x^{(i)})}\Delta x^{(i)} = 0 \tag{7.30}$$

where

$$JAC^{(x^{(i)})} = \left[\frac{\partial g(x)}{\partial x}\right]_{x=x^{(i)}} = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n(x)}{\partial x_1} & \cdots & \frac{\partial g_n(x)}{\partial x_n} \end{bmatrix}_{x=x^{(i)}} \tag{7.31}$$

is called the jacobian of g.

From (7.30), $\Delta x^{(i)}$ can be calculated as follows

$$\begin{aligned} JAC^{(x^{(i)})}\Delta x^{(i)} &= 0 - g(x^{(i)}) = \Delta g(x^{(i)}) \quad \Rightarrow & (7.32) \\ \Delta x^{(i)} &= \left[JAC^{(x^{(i)})}\right]^{-1} \Delta g(x^{(i)}) & (7.33) \end{aligned}$$

Since $g(x^{(i)}) = f(x^{(i)}) - b$, $\Delta g(x^{(i)})$ is given by

$$\Delta g(x^{(i)}) = b - f(x^{(i)}) = -g(x^{(i)}) \tag{7.34}$$

Furthermore, since b is constant, $JAC^{(x^{(i)})}$ is given by

$$JAC^{(x^{(i)})} = \left[\frac{\partial g(x)}{\partial x}\right]_{x=x^{(i)}} = \left[\frac{\partial f(x)}{\partial x}\right]_{x=x^{(i)}} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}_{x=x^{(i)}} \tag{7.35}$$

Therefore, $\Delta x^{(i)}$ can be calculated as follows

$$\Delta x^{(i)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \vdots \\ \Delta x_n^{(i)} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}_{x=x^{(i)}}^{-1} \begin{bmatrix} b_1 - f_1(x_1^{(i)}, \cdots, x_n^{(i)}) \\ \vdots \\ b_n - f_n(x_1^{(i)}, \cdots, x_n^{(i)}) \end{bmatrix} \tag{7.36}$$

Finally, the following is obtained

$$\begin{aligned} i &= i + 1 \\ x^{(i)} &= x^{(i-1)} + \Delta x^{(i-1)} \end{aligned}$$

The intention is that $x^{(1)}$ will estimate the solution x^* better than what $x^{(0)}$ does. In the same manner, $x^{(2)}$, $x^{(3)}$, $\cdots$ can be determined until a specified condition is satisfied. Thus, we obtain an iterative method according to the flowchart in Figure 7.9.

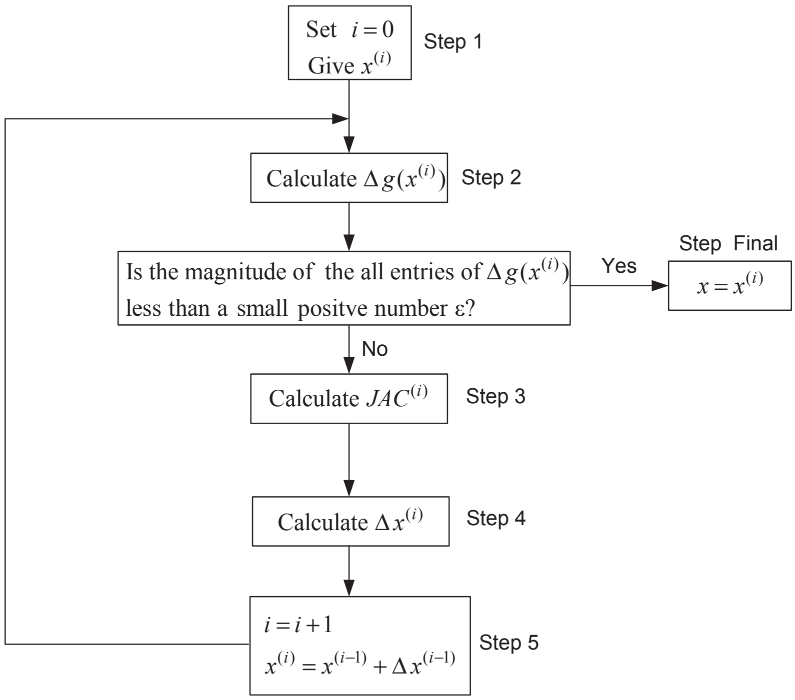


Figure 7.9. Flowchart for the Newton-Raphson method.

Example 7.10 *Using the Newton-Raphson method, solve for x of the equation*

$$g(x) = k_1\, x + k_2\, \cos(x - k_3) - k_4 = 0$$

Let $k_1 = -0.2$, $k_2 = 1.2$, $k_3 = -0.07$, $k_4 = 0.4$ *and* $\epsilon = 10^{-4}$.

Solution

This equation is of the form given by (7.28), with $f(x) = k_1\, x + k_2\, \cos(x - k_3)$ and $b = k_4$.

Step 1

Set $i = 0$ and $x^{(i)} = x^{(0)} = 0.0524$ (radians), i.e. 3 (degrees).

Step 2

$\Delta g(x^{(i)}) = b - f(x^{(i)}) = 0.4 - [(-0.2 * 0.0524) + 1.2\, \cos(0.0524 + 0.07)] = -0.7806$

Go to **Step 3** since $|\Delta g(x^{(i)})| > \epsilon$

Step 3

$JAC^{(x^{(i)})} = \left[\frac{\partial f}{\partial x}\right]_{x=x^{(i)}} = -0.2 - 1.2\, \sin(0.0524 + 0.07) = -0.3465$

Step 4

$\Delta x^{(i)} = \left[JAC^{(x^{(i)})}\right]^{-1}\, \Delta g(x^{(i)}) = \frac{-0.7806}{-0.3465} = 2.2529$

Step 5

$i = i + 1 = 0 + 1 = 1$
$x^{(i)} = x^{(i-1)} + \Delta x^{(i-1)} = 0.0524 + 2.2529 = 2.3053$. Go to **Step 2**

After 5 iterations, i.e. $i = 5$, it was found that $|\Delta g(x^{(i)})| < \epsilon$ for $x^{(5)} = 0.9809$ (rad.). Therefore, the solution becomes $x = 0.9809$ (rad.) or $x = 56.2000$ (deg.).

MATLAB-codes for this example can be found in Appendix A.

Comments on Example 7.10

Figure 7.10 shows variations of $g(x)$ versus x. The figure shows that the system (or equation) has only three solutions, i.e. the points at which $g(x) = 0$. Due to practical issues, x^* indicted with (O) in the figure is the interesting solution.

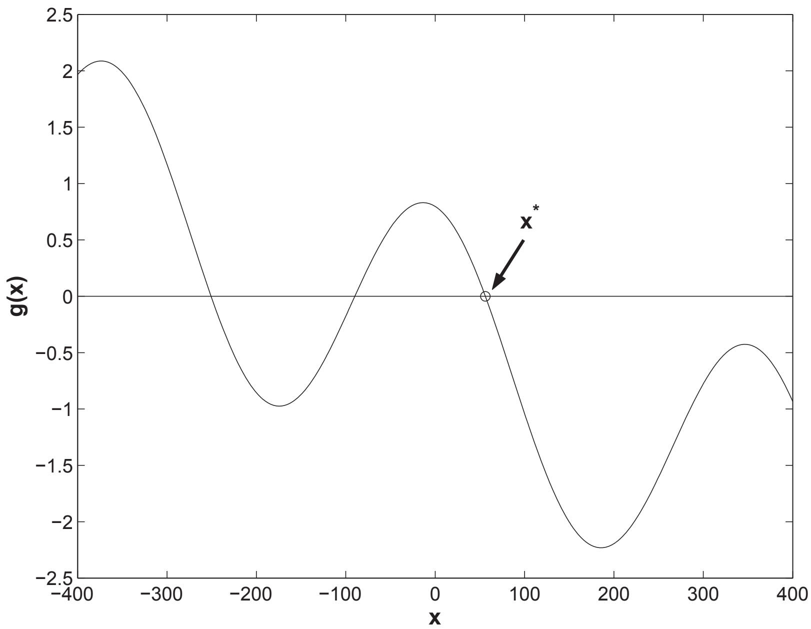


Figure 7.10. Variations of $g(x)$ vs. x.

Figure 7.11 shows how the equation is solved by the Newton-Raphson method.

We first guess the initial estimate $x^{(0)}$. In this case $x^{(0)} = 0.0524$ (rad.), i.e 3 (deg.). The tangent to $g(x)$ through the point $\left(x^{(0)}\ ,\ g(x^{(0)})\right)$, i.e. $g'(x^{(0)}) = \left[\frac{dg(x)}{dx}\right]_{x=x^{(0)}} = JAC^{(x^{(0)})}$, intersects the x–axis at point $x^{(1)}$. The equation for this tangent is given by

$$\mathcal{Y} - g(x^{(0)}) = g'(x^{(0)}) * (x - x^{(0)})$$

The intersection point $x^{(1)}$ is obtained by setting $\mathcal{Y} = 0$, i.e.

$$\begin{aligned} x^{(1)} &= x^{(0)} - \frac{g(x^{(0)})}{g'(x^{(0)})} = x^{(0)} - \left(g'(x^{(0)})\right)^{-1} g(x^{(0)}) \\ \Delta x^{(0)} &= x^{(1)} - x^{(0)} = -\left(g'(x^{(0)})\right)^{-1} g(x^{(0)}) = \left[JAC^{(x^{(0)})}\right]^{-1} \Delta g(x^{(0)}) \end{aligned}$$

In a similar manner, $x^{(2)}$ can be obtained which is hopefully a better estimate than $x^{(1)}$. As shown in the figure, from $x^{(2)}$ we obtain $x^{(3)}$ which is a better estimate of x^* than what $x^{(2)}$ does. This iterative method will be continued until $|\Delta g(x)| < \epsilon$.

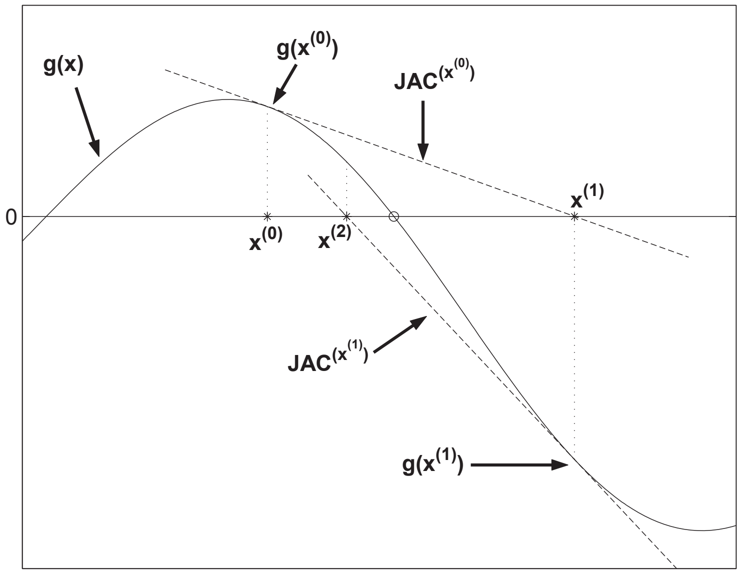


Figure 7.11. Variations of $g(x)$ vs. x.

Example 7.11 *Solve for* x *in Example 7.10, but let* $x^{(0)} = 0.0174$ *(rad.), i.e. 1 (deg.).*

Solution

D.I.Y, (i.e., Do It Yourself)

7.4.2 Application to power systems

Consider a power system with N buses. The aim is to determine the voltage at all buses in the system by applying the Newton-Raphson method. **All variables are expressed in pu**.

Consider again Figure 7.1. Let

$$\begin{aligned} g_{kj} + j\, b_{kj} &= \frac{1}{\overline{Z}_{kj}} = \frac{1}{R_{kj} + j\, X_{kj}} = \frac{R_{kj}}{Z_{kj}^2} + j\, \frac{-X_{kj}}{Z_{kj}^2} \quad \Rightarrow \\ g_{kj} &= \frac{R_{kj}}{Z_{kj}^2} \\ b_{kj} &= -\frac{X_{kj}}{Z_{kj}^2} \end{aligned} \tag{7.37}$$

Based on (7.37), we rewrite (7.3) and (7.4) as follows

$$P_{kj} = g_{kj}\, U_k^2 - U_k\, U_j\, [g_{kj} \cos(\theta_{kj}) + b_{kj} \sin(\theta_{kj})] \tag{7.38}$$

$$Q_{kj} = U_k^2(-b_{sh-kj} - b_{kj}) - U_k\, U_j\, [g_{kj} \sin(\theta_{kj}) - b_{kj} \cos(\theta_{kj})] \tag{7.39}$$

The current through the line, and the losses in the line can be calculated by

$$\overline{I}_{kj} = \frac{P_{kj} - j\,Q_{kj}}{\overline{U}_k^*} \tag{7.40}$$

$$P_{lkj} = P_{kj} + P_{jk} \tag{7.41}$$

$$Q_{lkj} = Q_{kj} + Q_{jk} \tag{7.42}$$

Consider again Figure 7.4. Let $Y = G + jB$ denote the admittance matrix of the system (or Y-matrix), where Y is an $N \times N$ matrix, i.e. the system has N buses. The relation between the injected currents into the buses and the voltages at the buses is given by $I = Y\,U$, see section 4.1. Therefore, the injected current into bus k is given by $\overline{I}_k = \sum_{j=1}^{N} \overline{Y}_{kj}\,\overline{U}_j$.

The injected complex power into bus k can now be calculated by

$$\overline{S}_k = \overline{U}_k\,\overline{I}_k^* = \overline{U}_k \sum_{j=1}^{N} \overline{Y}_{kj}^* \overline{U}_j^* = U_k \sum_{j=1}^{N} (G_{kj} - jB_{kj})\,U_j(\cos(\theta_{kj}) + j\sin(\theta_{kj}))$$

$$= \left(U_k \sum_{j=1}^{N} U_j\,[G_{kj}\,\cos(\theta_{kj}) + B_{kj}\,\sin(\theta_{kj})] \right) + j \left(U_k \sum_{j=1}^{N} U_j\,[G_{kj}\,\sin(\theta_{kj}) - B_{kj}\,\cos(\theta_{kj})] \right)$$

Let P_k denote the real part of $\overline{S}_k$, i.e. the injected active power, and Q_k denote the imaginary part of $\overline{S}_k$, i.e. the injected reactive power, as follows:

$$\begin{aligned} P_k &= U_k \sum_{j=1}^{N} U_j\,[G_{kj}\,\cos(\theta_{kj}) + B_{kj}\,\sin(\theta_{kj})] \\ Q_k &= U_k \sum_{j=1}^{N} U_j\,[G_{kj}\,\sin(\theta_{kj}) - B_{kj}\,\cos(\theta_{kj})] \end{aligned} \tag{7.43}$$

Note that $G_{kj} = -g_{kj}$ and $B_{kj} = -b_{kj}$ for $k \neq j$. Furthermore,

$$P_k = \sum_{j=1}^{N} P_{kj}$$

$$Q_k = \sum_{j=1}^{N} Q_{kj}$$

Equations (7.18) and (7.19) can now be rewritten as

$$\begin{aligned} P_k - P_{GDk} &= 0 \\ Q_k - Q_{GDk} &= 0 \end{aligned} \tag{7.44}$$

which are of the form given in equation (7.28), where

$$x = \begin{bmatrix} \theta \\ \\ U \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \\ \\ U_1 \\ \vdots \\ U_N \end{bmatrix}, \; f(\theta, U) = \begin{bmatrix} f_P(\theta, U) \\ \\ f_Q(\theta, U) \end{bmatrix} = \begin{bmatrix} P_1 \\ \vdots \\ P_N \\ \\ Q_1 \\ \vdots \\ Q_N \end{bmatrix}, \; b = \begin{bmatrix} b_P \\ \\ b_Q \end{bmatrix} = \begin{bmatrix} P_{GD1} \\ \vdots \\ P_{GDN} \\ \\ Q_{GD1} \\ \vdots \\ Q_{GDN} \end{bmatrix} \tag{7.45}$$

the aim is to determine $x = [\theta \quad U]^T$ by applying the Newton-Raphson method.

Assume that there are 1 slack bus and M PU-buses in the system. Therefore, θ becomes an $(N-1) \times 1$ vector and U becomes an $(N-1-M) \times 1$ vector, why?

Based on (7.34), we define the following:

$$\begin{aligned} \Delta P_k &= P_{GDk} - P_k \quad k \neq \text{slack bus} \\ \Delta Q_k &= Q_{GDk} - Q_k \quad k \neq \text{slack bus and PU-bus} \end{aligned} \tag{7.46}$$

Based on (7.35), the jacobian matrix is given by

$$JAC = \begin{bmatrix} \frac{\partial f_P(\theta,U)}{\partial \theta} & \frac{\partial f_P(\theta,U)}{\partial U} \\ \\ \frac{\partial f_Q(\theta,U)}{\partial \theta} & \frac{\partial f_Q(\theta,U)}{\partial U} \end{bmatrix} = \begin{bmatrix} H & N' \\ \\ J & L' \end{bmatrix} \tag{7.47}$$

where,

$$\begin{array}{llll} H & \text{is an} & (N-1) \times (N-1) & \text{matrix} \\ N' & \text{is an} & (N-1) \times (N-M-1) & \text{matrix} \\ J & \text{is an} & (N-M-1) \times (N-1) & \text{matrix} \\ L' & \text{is an} & (N-M-1) \times (N-M-1) & \text{matrix} \end{array}$$

The entries of these matrices are given by:

$$\begin{array}{lll} H_{kj} = \frac{\partial P_k}{\partial \theta_j} & k \neq \text{slack bus} & j \neq \text{slack bus} \\ \\ N'_{kj} = \frac{\partial P_k}{\partial U_j} & k \neq \text{slack bus} & j \neq \text{slack bus and PU-bus} \\ \\ J_{kj} = \frac{\partial Q_k}{\partial \theta_j} & k \neq \text{slack bus and PU-bus} & j \neq \text{slack bus} \\ \\ L'_{kj} = \frac{\partial Q_k}{\partial U_j} & k \neq \text{slack bus and PU-bus} & j \neq \text{slack bus and PU-bus} \end{array}$$

Based on (7.32), (7.46) and (7.47), the following is obtained

$$\begin{bmatrix} H & N' \\ J & L' \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta U \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} \tag{7.48}$$

To simplify the entries of the matrices N' and L', these matrices are multiplied with U. Then, (7.48) can be rewritten as

$$\begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \frac{\Delta U}{U} \end{bmatrix} = \begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta U' \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} \tag{7.49}$$

where,
for $k \neq j$

$$
\begin{aligned}
H_{kj} = & \quad \frac{\partial P_k}{\partial \theta_j} && = U_k\, U_j\, [G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj})] \\
N_{kj} = & \quad U_j\, N'_{kj} = U_j \frac{\partial P_k}{\partial U_j} && = U_k\, U_j\, [G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj})] \\
J_{kj} = & \quad \frac{\partial Q_k}{\partial \theta_j} && = -U_k\, U_j\, [G_{kj} \cos(\theta_{kj}) + B_{kj} \sin(\theta_{kj})] \\
L_{kj} = & \quad U_j\, L'_{kj} = U_j \frac{\partial Q_k}{\partial U_j} && = U_k\, U_j\, [G_{kj} \sin(\theta_{kj}) - B_{kj} \cos(\theta_{kj})]
\end{aligned}
\tag{7.50}
$$

and for $k = j$

$$
\begin{aligned}
H_{kk} = & \quad \frac{\partial P_k}{\partial \theta_k} && = -Q_k - B_{kk} U_k^2 \\
N_{kk} = & \quad U_k \frac{\partial P_k}{\partial U_k} && = P_k + G_{kk} U_k^2 \\
J_{kk} = & \quad \frac{\partial Q_k}{\partial \theta_k} && = P_k - G_{kk} U_k^2 \\
L_{kj} = & \quad U_k \frac{\partial Q_k}{\partial U_k} && = Q_k - B_{kk} U_k^2
\end{aligned}
\tag{7.51}
$$

Now based on (7.36), the following is obtained:

$$
\begin{bmatrix} \Delta\theta \\ \frac{\Delta U}{U} \end{bmatrix} = \begin{bmatrix} \Delta\theta \\ \Delta U' \end{bmatrix} = \begin{bmatrix} H & N \\ J & L \end{bmatrix}^{-1} \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}
\tag{7.52}
$$

Finally, U and θ will be updated as follows:

$$
\begin{array}{ll}
\theta_k = \theta_k + \Delta\theta_k & k \neq \text{slack bus} \\
U_k = U_k\,(1 + \Delta U'_k) & k \neq \text{slack bus and PU-bus}
\end{array}
\tag{7.53}
$$

7.4.3 Newton-Raphson method for solving power flow equations

Newton-Raphson method can be applied to non-linear power flow equations as follows:

- **Step 1**

  1a) Read bus and line data. Identify slack bus (i.e. Uθ-bus), PU-buses and PQ-buses.

  1b) Build the Y-matrix and calculate the net productions, i.e. $P_{GD} = P_G - P_{LD}$ and $Q_{GD} = Q_G - Q_{LD}$.

  1c) Give the initial estimate of the unknown variables, i.e. U for PQ-buses and θ for PU- and PQ-buses. It is very common to set $U = U_{slack}$ and $\theta = \theta_{slack}$. However, the flat initial estimate may also be applied, i.e. $U = 1$ and $\theta = 0$.

  1d) Go to **Step 2**.

- **Step 2**

2a) Calculate the injected power into each bus by equation (7.43).

2b) Calculate the difference between the net production and the injected power for each bus, i.e. ΔP and ΔQ by equation (7.46).

2c) Is the magnitude of all entries of $[\Delta P \quad \Delta Q]^T$ less than a specified small positive constant ϵ ?

  * If yes, go to **Step Final**.
  * if no, go to **Step 3**.

- **Step 3**

3a) Calculate the jacobian by equations (7.50) and (7.51).

3b) Go to **Step 4**.

- **Step 4**

4a) Calculate $[\Delta\theta \quad \Delta U']^T$ by equation (7.52).

4b) Go to **Step 5**.

- **Step 5**

5a) Update U and θ by equation (7.53).

5b) Go till **Step 2**.

- **Step Final**

  - Calculate the generated powers, i.e. P_G (MW) and Q_G (MVAr) in the slack bus, and Q_G (MVAr) in the PU-buses by using equation (7.44).
  - Calculate the power flows (MW, MVAr) by using equations (7.38) and (7.39).
  - Calculate active power losses (MW) by using equation (7.41).
  - Give all the voltage magnitudes (kV) and the voltage phase angles (degrees).
  - Print out the results.

Example 7.12 *Consider the power system shown in Figure 7.12. Let* $S_{base} = 100$ *MVA, and* $U_{base} = 220$ *kV.*

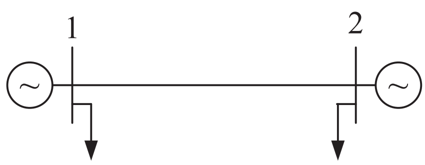


Figure 7.12. Single-line diagram of a balanced two-bus power system.

The following data (all in pu) is known:

- **Line between Bus 1 and Bus 2:** *short line,* $\bar{Z}_{12} = 0.02 + j\,0.2$

- **Bus 1:** *slack bus,* $U_1 = 1$, $\theta_1 = 0$, $P_{LD1} = 0.2$, $Q_{LD1} = 0.02$
- **Bus 2:** *PU-bus,* $U_2 = 1$, $P_{G2} = 1$, $P_{LD2} = 2$, $Q_{LD2} = 0.2$

By applying Newton-Raphson method, calculate θ_2, P_{G1}, Q_{G1}, Q_{G2} *and the active power losses in the system after 3 iterations.*

Solution

MATLAB-codes for this example can be found in Appendix A.

Step 1
1a) bus 1 is a slack bus , bus 2 is a PU-bus , $U_1 = 1$, $U_2 = 1$, $\theta_1 = 0$.

1b)

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{Z_{12}} & -\frac{1}{Z_{12}} \\ -\frac{1}{Z_{12}} & \frac{1}{Z_{12}} \end{bmatrix} = \begin{bmatrix} G_{11} + j\,B_{11} & G_{12} + j\,B_{12} \\ G_{21} + j\,B_{21} & G_{22} + j\,B_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} 0.4950 - j\,4.9505 & -0.4950 + j\,4.9505 \\ -0.4950 + j\,4.9505 & 0.4950 - j\,4.9505 \end{bmatrix} = \mathbf{G} + j\,\mathbf{B}$$

$P_{GD2} = P_{G2} - P_{LD2} = 1 - 2 = -1$

No Q_{GD} since there is no PQ-bus in the system.

1c)
Since the system has only one slack bus and one PU-bus, the phase angle of the PU-bus is the only unknown variable. As an initial value , let $\theta_2 = 0$.

Iteration 1

Step 2
2a)

$$\begin{aligned} P_2 &= U_2\,U_1\,[G_{21}\,\cos(\theta_2 - \theta_1) + B_{21}\,\sin(\theta_2 - \theta_1)] + U_2^2\,G_{22} = \\ &= 1 * 1 * [-0.4950 * \cos(0-0) + 4.9505 * \sin(0-0)] + 1^2 * 0.4950 = 0 \end{aligned}$$

2b)

$$\Delta P = \Delta P_2 = P_{GD2} - P_2 = -1 - 0 = -1$$

Step 3

$$\begin{aligned} Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2 - \theta_1) - B_{21}\,\cos(\theta_2 - \theta_1)] - U_2^2\,B_{22} = \\ &= 1 * 1 * [-0.4950 * \sin(0-0) - 4.9505 * \cos(0-0)] - 1^2 * (-4.9505) = 0 \\ H &= \frac{\partial P_2}{\partial \theta_2} = -Q_2 - B_{22}U_2^2 = -0 - (-4.9505 * 1^2) = 4.9505 \\ JAC &= \begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} 4.9505 \end{bmatrix} \end{aligned}$$

Step 4

$$\Delta\theta_2 = H^{-1}\,\Delta P_2 = \frac{-1}{4.9505} = -0.2020$$

Step 5

$$\theta_2 = \theta_2 + \Delta\theta_2 = 0 - 0.2020 = -0.2020$$

Iteration 2
Step 2
2a)

$$\begin{aligned} P_2 &= U_2\,U_1\,[G_{21}\,\cos(\theta_2-\theta_1) + B_{21}\,\sin(\theta_2-\theta_1)] + U_2^2\,G_{22} = \\ &= 1*1*[-0.4950*\cos(-0.2020-0) + 4.9505*\sin(-0.2020-0)] + 1^2*0.4950 = -0.9831 \end{aligned}$$

2b)

$$\Delta P = \Delta P_2 = P_{GD2} - P_2 = -1 - (-0.9831) = -0.0169$$

Step 3

$$\begin{aligned} Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2-\theta_1) - B_{21}\,\cos(\theta_2-\theta_1)] - U_2^2\,B_{22} = \\ &= 1*1*[-0.4950*\sin(-0.2020-0) - 4.9505*\cos(-0.2020-0)] - 1^2*(-4.9505) = 0.2000 \\ H &= \frac{\partial P_2}{\partial\theta_2} = -Q_2 - B_{22}U_2^2 = -0.2000 - (-4.9505*1^2) = 4.7505 \\ JAC &= \big[H\big] = \big[4.7505\big] \end{aligned}$$

Step 4

$$\Delta\theta_2 = H^{-1}\,\Delta P_2 = \frac{-0.0169}{4.7505} = -0.0035$$

Step 5

$$\theta_2 = \theta_2 + \Delta\theta_2 = -0.2020 - 0.0035 = -0.2055$$

Iteration 3
Step 2
2a)

$$\begin{aligned} P_2 &= U_2\,U_1\,[G_{21}\,\cos(\theta_2-\theta_1) + B_{21}\,\sin(\theta_2-\theta_1)] + U_2^2\,G_{22} = \\ &= 1*1*[-0.4950*\cos(-0.2055-0) + 4.9505*\sin(-0.2055-0)] + 1^2*0.4950 = -1.0000 \end{aligned}$$

2b)

$$\Delta P = \Delta P_2 = P_{GD2} - P_2 = -1 - (-1.0000) \approx 0 \quad (\text{in MATLAB } \Delta P_2 = -9.3368*10^{-6})$$

Step 3

$$\begin{aligned} Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2-\theta_1) - B_{21}\,\cos(\theta_2-\theta_1)] - U_2^2\,B_{22} = \\ &= 1*1*[-0.4950*\sin(-0.2055-0) - 4.9505*\cos(-0.2055-0)] - 1^2*(-4.9505) = 0.2053 \\ H &= \frac{\partial P_2}{\partial\theta_2} = -Q_2 - B_{22}U_2^2 = -0.2053 - (-4.9505*1^2) = 4.7452 \\ JAC &= \big[H\big] = \big[4.7452\big] \end{aligned}$$

Step 4

$$\Delta\theta_2 = H^{-1}\,\Delta P_2 = \frac{-9.3368 * 10^{-6}}{4.7452} = -1.9676 * 10^{-6} \approx 0$$

Step 5

$$\theta_2 = \theta_2 + 0 = -0.2055 - 0 = -0.2055$$

Now go to **Step Final**

Step Final

$$\begin{aligned}
P_1 &= U_1\,U_2\,[G_{12}\,\cos(\theta_1-\theta_2) + B_{12}\,\sin(\theta_1-\theta_2)] + U_1^2\,G_{11} = \\
&= 1*1*[-0.4950*\cos(0+0.2055) + 4.9505*\sin(0+0.2055)] + 1^2*0.4950 = 1.0208 \\
Q_1 &= U_1\,U_2\,[G_{12}\,\sin(\theta_1-\theta_2) - B_{12}\,\cos(\theta_1-\theta_2)] - U_1^2\,B_{11} = \\
&= 1*1*[-0.4950*\sin(0+0.2055) - 4.9505*\cos(0+0.2055)] - 1^2*(-4.9505) = 0.0032 \\
Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2-\theta_1) - B_{21}\,\cos(\theta_2-\theta_1)] - U_2^2\,B_{22} = \\
&= 1*1*[-0.4950*\sin(-0.2055-0) - 4.9505*\cos(-0.2055-0)] - 1^2*(-4.9505) = 0.2053 \\
P_{G1} &= (P_1 + P_{LD1})*S_{base} = (1.0208+0.2)*100 = 122.08 \quad \text{MW} \quad (\text{in MATLAB } P_{G1}{=}122.0843) \\
Q_{G1} &= (Q_1 + Q_{LD1})*S_{base} = (0.0032+0.02)*100 = 2.32 \quad \text{MVAr} \quad (\text{in MATLAB } Q_{G1}{=}2.3171) \\
Q_{G2} &= (Q_2 + Q_{LD2})*S_{base} = (0.2053+0.2)*100 = 40.53 \quad \text{MVAr} \quad (\text{in MATLAB } Q_{G2}{=}40.5255)
\end{aligned}$$

$$\begin{aligned}
g &= -G \quad , \quad b = -B \quad \text{and} \quad b_{sh-12} = 0 \\
P_{12} &= \big(g_{12}\,U_1^2 - U_1\,U_2\,[g_{12}\,\cos(\theta_1-\theta_2) + b_{12}\,\sin(\theta_1-\theta_2)]\big)*S_{base} = \\
&= \big(0.4950*1^2 - 1*1*[0.4950*\cos(0+0.2055) - 4.9505*\sin(0+0.2055)]\big)*100 = \\
&= 102.0843 \quad \text{MW} \\
P_{21} &= \big(g_{21}\,U_2^2 - U_2\,U_1\,[g_{21}\,\cos(\theta_2-\theta_1) + b_{21}\,\sin(\theta_2-\theta_1)]\big)*S_{base} = \\
&= \big(0.4950*1^2 - 1*1*[0.4950*\cos(-0.2055-0) - 4.9505*\sin(-0.2055-0)]\big)*100 = \\
&= -100 \quad \text{MW} \\
Q_{12} &= \big((-b_{sh-12} - b_{12})U_1^2 - U_1\,U_2\,[g_{12}\,\sin(\theta_1-\theta_2) - b_{12}\,\cos(\theta_1-\theta_2)]\big)*S_{base} = \\
&= \big((-0+4.9505)*1^2 - 1*1*[0.4950*\sin(0+0.2055) + 4.9505*\cos(0+0.2055)]\big)*100 = \\
&= 0.3171 \quad \text{MVAr} \\
Q_{21} &= \big((-b_{sh-12} - b_{21})U_2^2 - U_2\,U_1\,[g_{21}\,\sin(\theta_2-\theta_1) - b_{21}\,\cos(\theta_2-\theta_1)]\big)*S_{base} = \\
&= \big((-0+4.9505)*1^2 - 1*1*[0.4950*\sin(-0.2055-0) + 4.9505*\cos(-0.2055-0)]\big)*100 \\
&= 20.5255 \quad \text{MVAr}
\end{aligned}$$

$$\begin{aligned}
P_{Loss}^{tot} &= P_{12} + P_{21} = 102.0843 - 100 = 2.0843 \quad \text{MW} \\
\text{or}& \\
P_{Loss}^{tot} &= (P_{G1} + P_{G2}) - (P_{LD1} + P_{LD2}) = (122.0843 + 100) - (20 + 200) = 2.0843 \quad \text{MW}
\end{aligned}$$

$$\begin{aligned}
\text{ANG} &= [\theta_1 \quad \theta_2]*\frac{180}{\pi} = [0 \quad -0.2055]*\frac{180}{\pi} = [0 \quad -11.7771^\circ] \\
\text{VOLT} &= [U_1 \quad U_2]*U_{base} = [1 \quad 1]*220 = [220 \quad 220]
\end{aligned}$$

Example 7.13 *Consider again the power system shown in Figure 7.12. In this example, let bus 2 be a PQ-bus with the following data:*

- **Bus 2:** *PQ-bus,* $P_{G2} = 1$, $Q_{G2} = 0.405255$, $P_{LD2} = 2$, $Q_{LD2} = 0.2$

Let also $\varepsilon = 10^{-6}$*. By applying Newton-Raphson method, calculate* θ_2*,* P_{G1}*,* Q_{G1}*,* Q_{G2} *and the active power losses.*

Solution

See the MATLAB-codes in Appendix A.

Step 1
1a) bus 1 is a slack bus , bus 2 is a PQ-bus , $U_1 = 1$ and $\theta_1 = 0$.

1b)

$$\begin{aligned}
\mathbf{Y} &= \begin{bmatrix} G_{11} + j\,B_{11} & G_{12} + j\,B_{12} \\ \\ G_{21} + j\,B_{21} & G_{22} + j\,B_{22} \end{bmatrix} = \mathbf{G} + j\,\mathbf{B} \\
P_{GD2} &= P_{G2} - P_{LD2} \\
Q_{GD2} &= Q_{G2} - Q_{LD2}
\end{aligned}$$

1c)
Since the system has only one PQ-bus, the voltage and phase angle of bus 2 are the unknown variables. As an initial value , let $\theta_2 = 0$ and $U_2 = 1$.

Step 2
2a)

$$\begin{aligned}
P_2 &= U_2\,U_1\,[G_{21}\,\cos(\theta_2 - \theta_1) + B_{21}\,\sin(\theta_2 - \theta_1)] + U_2^2\,G_{22} \\
Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2 - \theta_1) - B_{21}\,\cos(\theta_2 - \theta_1)] - U_2^2\,B_{22}
\end{aligned}$$

2b)

$$\begin{aligned}
\Delta P &= \Delta P_2 = P_{GD2} - P_2 \\
\Delta Q &= \Delta Q_2 = Q_{GD2} - Q_2
\end{aligned}$$

Step 3
As long as $|\Delta P_2| > \varepsilon$ **and** $|\Delta Q_2| > \varepsilon$, perform **Step 3** as follows:

$$\begin{aligned}
H &= -Q_2 - B_{22}U_2^2 \\
N &= P_2 + G_{22}U_2^2 \\
J &= P_2 - G_{22}U_2^2 \\
L &= Q_2 - B_{22}U_2^2 \\
JAC &= \begin{bmatrix} H & N \\ J & L \end{bmatrix} \\
\begin{bmatrix} \Delta\theta_2 \\ \Delta U_2' \end{bmatrix} &= JAC^{-1} \begin{bmatrix} \Delta P_2 \\ \Delta Q_2 \end{bmatrix}
\end{aligned}$$

Next, update ΔP_2 and ΔQ_2 based on updated θ_2 and U_2 (i.e **Steps 4-5**) as follows:

$$\begin{aligned}
\theta_2 &= \theta_2 + \Delta\theta_2 \\
U_2 &= U_2\,(1 + \Delta U_2') \\
P_2 &= U_2\,U_1\,[G_{21}\,\cos(\theta_2 - \theta_1) + B_{21}\,\sin(\theta_2 - \theta_1)] + U_2^2\,G_{22} \\
Q_2 &= U_2\,U_1\,[G_{21}\,\sin(\theta_2 - \theta_1) - B_{21}\,\cos(\theta_2 - \theta_1)] - U_2^2\,B_{22} \\
\Delta P &= \Delta P_2 = P_{GD2} - P_2 \\
\Delta Q &= \Delta Q_2 = Q_{GD2} - Q_2
\end{aligned}$$

and check if $|\Delta P_2| < \varepsilon$ **and** $|\Delta Q_2| < \varepsilon$.

Step Final

See **Step Final** in Example 7.12.

In the next examples, it will be shown that how the "fsolve" function in MATLAB can be used for solving non-linear power flow equations.

Example 7.14 *Consider the power system shown in Figure 7.13. Let the base power be $S_{base} = 100$ MVA, the base voltage be $U_{base} = 220$ kV. Let also, bus 1 be a slack bus.*

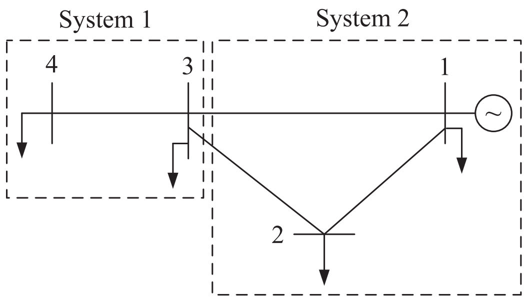


Figure 7.13. Single-line diagram of a balanced four-bus power system.

The system data (in MW, MVAr, kV, Ω and S) is given as follows:

- **Line between Bus 1 and Bus 2:** $\bar{Z}_{12} = 5 + j\,65$, $b_{sh-12} = 0.0002$
- **Line between Bus 1 and Bus 3:** $\bar{Z}_{13} = 4 + j\,60$, $b_{sh-13} = 0.0002$
- **Line between Bus 2 and Bus 3:** $\bar{Z}_{23} = 5 + j\,68$, $b_{sh-23} = 0.0002$
- **Line between Bus 3 and Bus 4:** $\bar{Z}_{34} = 3 + j\,30$, *short line*
- **Bus 1:** $U_1 = 220$, $\theta_1 = 0$, $P_{LD1} = 10$, $Q_{LD1} = 2$
- **Bus 2:** $P_{LD2} = 90$, $Q_{LD2} = 10$
- **Bus 3:** $P_{LD3} = 80$, $Q_{LD3} = 10$
- **Bus 4:** $P_{LD4} = 50$, $Q_{LD4} = 10$

Use "fsolve" function in MATLAB, and find

a) *the unknown voltage magnitudes and voltage phase angles,*

b) *the generated active and reactive powers at the slack bus, and the generated reactive powers at PU-buses (if any),*

c) *the total active power losses, and the losses in System1 and System 2,*

d) *the changes (in % compared to the obtained results in task* **c)***) of power losses in both systems, for an active load increased at bus 2 with* 30 *MW, i.e.* $P_{LD2}^{new} = 120$ *MW.*

e) *Let* $P_{LD2} = 90$ *MW. Re-do task* **d)** *for a reactive load increased at bus 3 with* 10 *MVAr, i.e.* $Q_{LD3}^{new} = 20$ *MVAr.*

Solution

MATLAB-codes for this example can be found in Appendix A.

For using "fsolve", you may need two MATLAB-files, one main file and one for solving x in $0 = g(x)$ by "fsolve" function, (see the MATLAB-codes in Appendix A). In the second file you need to define x and $g(x)$ as follows:

$$
\begin{array}{lcll}
x & = & [\theta_2\, \theta_3\, \theta_4\, U_2\, U_3\, U_4] & \\
0 & = & g_1(x) = P_2 - P_{GD2} & , \quad \text{(active power mismatch at bus 2, see equation (7.44))} \\
0 & = & g_2(x) = P_3 - P_{GD3} & , \quad \text{(active power mismatch at bus 3)} \\
0 & = & g_3(x) = P_4 - P_{GD4} & , \quad \text{(active power mismatch at bus 4)} \\
0 & = & g_4(x) = Q_2 - Q_{GD2} & , \quad \text{(reactive power mismatch at bus 2)} \\
0 & = & g_5(x) = Q_3 - Q_{GD3} & , \quad \text{(reactive power mismatch at bus 3)} \\
0 & = & g_6(x) = Q_4 - Q_{GD4} & , \quad \text{(reactive power mismatch at bus 4)}
\end{array}
$$

where, P_k and Q_k can be obtained based on equation (7.43).

a) $U_1 = 1.0000 \times U_{base} = 220.0000$ kV, $\theta_1 = 0°$,
$U_2 = 0.9864 \times U_{base} = 216.9990$ kV, $\theta_2 = -7.8846°$,
$U_3 = 0.9794 \times U_{base} = 215.4704$ kV, $\theta_3 = -8.7252°$,
$U_4 = 0.9693 \times U_{base} = 213.2499$ kV, $\theta_4 = -10.5585°$,

b) Slack bus (bus 1): $P_{G1} = 232.4938$ MW, $Q_{G1} = 9.6185$ MVAr

c) $P_{Loss}^{\text{Sys1}} = 0.1715$ MW , $P_{Loss}^{\text{Sys2}} = 2.3222$ MW , $P_{Loss}^{\text{tot}} = 2.4938$ MW

d)
$$
\begin{array}{lcl}
P_{Loss}^{\text{Sys1}} = 0.1729 \text{ MW} & \Rightarrow & \Delta P_{Loss}^{\text{Sys1}} = 0.8163\,\% \\
P_{Loss}^{\text{Sys2}} = 3.0236 \text{ MW} & \Rightarrow & \Delta P_{Loss}^{\text{Sys2}} = 30.2041\,\% \\
P_{Loss}^{\text{tot}} = 3.1966 \text{ MW} & \Rightarrow & \Delta P_{Loss}^{\text{tot}} = 28.1830\,\%
\end{array}
$$

e) $P_{Loss}^{\text{Sys1}} = 0.1749$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys1}} = 1.9825\,\%$
$P_{Loss}^{\text{Sys2}} = 2.3629$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys2}} = 1.7526\,\%$
$P_{Loss}^{\text{tot}} = 2.5378$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{tot}} = 1.7685\,\%$

Example 7.15 *Consider again the power system in Example 7.14. The System 1 operator is interested in the results of the power flow calculations when installing a controllable shunt capacitor at bus 3 to keep the voltage at its rated (or nominal) value, i.e.* $U_3 = 220$ *kV. Re-do the tasks in Example 7.14, and also find the size of the shunt capacitor* B_{sh} *in S.*

Solution

In this example, bus 3 will be considered as a PU-bus with $U_3 = 220$ kV. MATLAB-codes for this example can be found in Appendix A. Note that only the changes of the MATLAB-codes compared to the MATLAB-codes for Example 7.14 are given.

a) $U_1 = 1.0000 \times U_{base} = 220.0000$ kV, $\theta_1 = 0°$,
$U_2 = 0.9968 \times U_{base} = 219.2882$ kV, $\theta_2 = -7.8192°$,
$U_3 = 1.0000 \times U_{base} = 220.0000$ kV, $\theta_3 = -8.6473°$,
$U_4 = 0.9901 \times U_{base} = 217.8306$ kV, $\theta_4 = -10.4051°$,

b) Slack bus (bus 1): $P_{G1} = 232.4490$ MW, $Q_{G1} = -14.7469$ MVAr
PU-buses (bus 3): $Q_{G3} = 22.5772$ MVAr and $B_{sh} = 0.00046647$ S

c) $P_{Loss}^{\text{Sys1}} = 0.1644$ MW , $P_{Loss}^{\text{Sys2}} = 2.2846$ MW , $P_{Loss}^{\text{tot}} = 2.4490$ MW

d) $P_{Loss}^{\text{Sys1}} = 0.1644$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys1}} = 0\,\%$
$P_{Loss}^{\text{Sys2}} = 2.9622$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys2}} = 29.6595\,\%$
$P_{Loss}^{\text{tot}} = 3.1266$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{tot}} = 27.6684\,\%$

e) $P_{Loss}^{\text{Sys1}} = 0.1644$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys1}} = 0\,\%$
$P_{Loss}^{\text{Sys2}} = 2.2846$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{Sys2}} = 0\,\%$
$P_{Loss}^{\text{tot}} = 2.4490$ MW $\Rightarrow$ $\Delta P_{Loss}^{\text{tot}} = 0\,\%$

Example 7.16 *Consider the power system described in Example 7.15. Now, both system operators are interested in the results of the power flow calculations when the generator at bus 1 has a fixed generation with* P_{G1} *and* Q_{G1} *obtained in Example 7.15, and a new generator is installed at bus 4 to be a slack bus with* U_4 *and* θ_4 *obtained in Example 7.15. Re-do the tasks in Example 7.15.*

Solution

In this example, bus 1 will be considered as a PQ-bus. After modifying the MATLAB-codes for Example 7.15, the load flow simulations give the following results:

a) $U_1 = 1.0000 \times U_{base} = 220.0000$ kV, $\theta_1 = 0°$,
$U_2 = 0.9968 \times U_{base} = 219.2882$ kV, $\theta_2 = -7.8192°$,
$U_3 = 1.0000 \times U_{base} = 220.0000$ kV, $\theta_3 = -8.6473°$,
$U_4 = 0.9901 \times U_{base} = 217.8306$ kV, $\theta_4 = -10.4051°$,

b) Slack bus (bus 4): $P_{G4} = 0$ MW, $Q_{G4} = 0$ MVAr
PU-buses (bus 3): $Q_{G3} = 22.5772$ MVAr and $B_{sh} = 0.00046647$ S

c) $P_{Loss}^{\text{Sys1}} = 0.1644$ MW , $P_{Loss}^{\text{Sys2}} = 2.2846$ MW , $P_{Loss}^{\text{tot}} = 2.4490$ MW

d) $P_{Loss}^{\text{Sys1}} = 0.0373 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{Sys1}} = -77.3114\,\%$
$P_{Loss}^{\text{Sys2}} = 2.3232 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{Sys2}} = 1.6896\,\%$
$P_{Loss}^{\text{tot}} = 2.3605 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{tot}} = -3.6137\,\%$

e) $P_{Loss}^{\text{Sys1}} = 0.1644 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{Sys1}} = 0\,\%$
$P_{Loss}^{\text{Sys2}} = 2.2846 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{Sys2}} = 0\,\%$
$P_{Loss}^{\text{tot}} = 2.4490 \text{ MW} \quad \Rightarrow \quad \Delta P_{Loss}^{\text{tot}} = 0\,\%$

Some questions regarding the obtained results:

q1: Why is $\Delta P_{Loss}^{\text{Sys1}} = 0$ in Example 7.15, task **d)**, but not in Example 7.14 and Example 7.16?

q2: Why is $\Delta P_{Loss}^{\text{tot}} = 0$ in Example 7.15 and Example 7.16 , task **e)**, but not in Example 7.14?

q3: Why have $\Delta P_{Loss}^{\text{Sys1}}$ and $\Delta P_{Loss}^{\text{tot}}$ in Example 7.16, task **d)**, decreased?

q4: In Example 7.15, does B_{sh} in tasks **d)** and **e)** have the same value as that obtained in task **b)**? Motivate your answer.

q5: Why are the obtained voltages in Example 7.15 and Example 7.16 identical?

Chapter 8
Symmetrical components

8.1 Definitions

Assume an arbitrary un-symmetric combination of three phases, exemplified by the currents $\overline{I}_a, \overline{I}_b$ and $\overline{I}_c$, as shown in Figure 8.1 a).

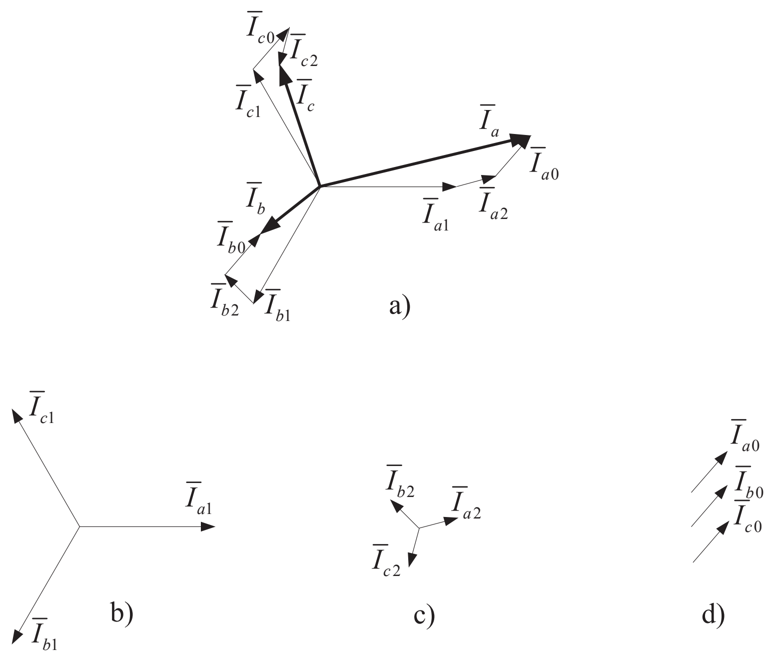


Figure 8.1. Unbalanced current phasors expressed as the sum of positive-, negative-, and zero-sequence components.

Based on C. L. Fortesque's theorem, a set of three unbalanced phasors in a three-phase system can be resolved into the following three balanced systems of phasors (or *symmetrical components*) :

A. *Positive-sequence components* consisting of a balanced system of three phasors with the same amplitude, and having a phase displacement of 120 and 240°, respectively. The phase sequence is *abca*, as shown in Figure 8.1 b).

B. *Negative-sequence components* consisting of a balanced system of three phasors with the same amplitude, and having a phase displacement of 240 and 120°, respectively. The phase sequence is *acba*, as shown in Figure 8.1 c).

C. *Zero-sequence components* consisting of a balanced system of three phasors with the same amplitude and phase, as shown in Figure 8.1 d).

The three balanced systems can be symbolized with 1 (positive-sequence), 2 (negative-sequence) and 0 (zero-sequence).

The result shown in Figure 8.1 can be mathematically expressed as :

$$\begin{aligned}
\overline{I}_a &= \overline{I}_{a1} + \overline{I}_{a2} + \overline{I}_{a0} \\
\overline{I}_b &= \overline{I}_{b1} + \overline{I}_{b2} + \overline{I}_{b0} \\
\overline{I}_c &= \overline{I}_{c1} + \overline{I}_{c2} + \overline{I}_{c0}
\end{aligned} \tag{8.1}$$

The three positive-sequence components can be denoted as

$$\begin{aligned}
\overline{I}_{b1} &= \overline{I}_{a1} e^{-j120^\circ} \\
\overline{I}_{c1} &= \overline{I}_{a1} e^{j120^\circ}
\end{aligned} \tag{8.2}$$

The corresponding expressions for the negative- and zero-sequence components are as

$$\begin{aligned}
\overline{I}_{b2} &= \overline{I}_{a2} e^{j120^\circ} \\
\overline{I}_{c2} &= \overline{I}_{a2} e^{-j120^\circ} \\
\overline{I}_{a0} &= \overline{I}_{b0} = \overline{I}_{c0}
\end{aligned} \tag{8.3}$$

By inserting equation (8.2) and (8.3) into equation (8.1), the following is obtained

$$\begin{aligned}
\overline{I}_a &= \overline{I}_{a1} + \overline{I}_{a2} + \overline{I}_{a0} \\
\overline{I}_b &= \alpha^2 \overline{I}_{a1} + \alpha \overline{I}_{a2} + \overline{I}_{a0} \\
\overline{I}_c &= \alpha \overline{I}_{a1} + \alpha^2 \overline{I}_{a2} + \overline{I}_{a0}
\end{aligned} \tag{8.4}$$

where,

$$\alpha = e^{j120^\circ} = \cos 120^\circ + j \sin 120^\circ = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \tag{8.5}$$

The following expressions of the symbol α are valid

$$\begin{aligned}
\alpha^2 = e^{j240^\circ} &= e^{-j120^\circ} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\
\alpha^3 &= 1 \\
1 + \alpha + \alpha^2 &= 0 \\
\alpha^* &= \alpha^2 \\
(\alpha^2)^* &= \alpha
\end{aligned} \tag{8.6}$$

Equation (8.4) can, by using matrix form, be written as

$$\mathbf{I_{Ph}} = \mathbf{T I_s} \tag{8.7}$$

where the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} \tag{8.8}$$

is called the *transformation matrix for the symmetrical components, (see also Appendix B where various linear transformations are presented).* The current vector

$$\mathbf{I_{Ph}} = \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} \tag{8.9}$$

represents the current phasor of each phase whereas

$$\mathbf{I_s} = \begin{bmatrix} \overline{I}_{a1} \\ \overline{I}_{a2} \\ \overline{I}_{a0} \end{bmatrix} \quad \text{or just} \quad \mathbf{I_s} = \begin{bmatrix} \overline{I}_{-1} \\ \overline{I}_{-2} \\ \overline{I}_{-0} \end{bmatrix} \tag{8.10}$$

represents the symmetrical components of the phase a current from which (based on equations (8.2)-(8.3)) the symmetrical components of the other phase currents can be obtained.

By using equation (8.7), the symmetrical components as a function of the phase currents can be obtained by :

$$\mathbf{I_s} = \mathbf{T^{-1}} \mathbf{I_{Ph}} \tag{8.11}$$

where

$$\mathbf{T^{-1}} = \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \tag{8.12}$$

Of course, the symmetrical components can also be applied to voltages. Using the vectors

$$\mathbf{U_{Ph}} = \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} \quad \text{and} \quad \mathbf{U_s} = \begin{bmatrix} \overline{U}_{a1} \\ \overline{U}_{a2} \\ \overline{U}_{a0} \end{bmatrix} \quad \text{or just} \quad \mathbf{U_s} = \begin{bmatrix} \overline{U}_{-1} \\ \overline{U}_{-2} \\ \overline{U}_{-0} \end{bmatrix} \tag{8.13}$$

for representing line-to-neutral voltage phasors and symmetrical components, respectively, the relation between them can be written as

$$\mathbf{U_{Ph}} = \mathbf{T} \mathbf{U_s} \tag{8.14}$$

$$\mathbf{U_s} = \mathbf{T^{-1}} \mathbf{U_{Ph}} \tag{8.15}$$

Example 8.1 *Calculate the symmetrical components for the following symmetrical voltages*

$$\mathbf{U_{Ph}} = \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \begin{bmatrix} 277\angle 0° \\ 277\angle - 120° \\ 277\angle + 120° \end{bmatrix} \text{ V} \tag{8.16}$$

Solution

By using equations (8.12) and (8.15), the symmetrical components of the voltage $\mathbf{U_{Ph}}$ can be calculated as

$$\begin{aligned} \begin{bmatrix} \overline{U}_{-1} \\ \overline{U}_{-2} \\ \overline{U}_{-0} \end{bmatrix} &= \mathbf{U_s} = \mathbf{T^{-1}} \mathbf{U_{Ph}} = \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \\ &= \frac{1}{3} \begin{bmatrix} 1 \cdot 277 + 1\angle 120° \cdot 277\angle - 120° + 1\angle 240° \cdot 277\angle + 120° \\ 1 \cdot 277 + 1\angle 240° \cdot 277\angle - 120° + 1\angle 120° \cdot 277\angle + 120° \\ 1 \cdot 277 + 1 \cdot 277\angle - 120° + 1 \cdot 277\angle + 120° \end{bmatrix} = \\ &= \begin{bmatrix} 277\angle 0° \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{8.17}$$

As given in the example, a symmetric three-phase system with a phase sequence of *abc* gives rise to a positive-sequence voltage only, having the same amplitude and angle as the voltage in phase a.

Example 8.2 *For a Y0-connected three-phase load (Y0 means that the neutral point is grounded either solidly or through an impedance), phase b is at one occasion disconnected. The load currents at that occasion are :*

$$\mathbf{I_{Ph}} = \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0 \\ 10\angle + 120^\circ \end{bmatrix} A \tag{8.18}$$

Calculate the symmetrical components of the load current as well as the current through the neutral-ground conductor, $\overline{I}_n$.

Solution

$$\begin{aligned} \begin{bmatrix} \overline{I}_{-1} \\ \overline{I}_{-2} \\ \overline{I}_{-0} \end{bmatrix} &= \frac{1}{3} \begin{bmatrix} 1 \cdot 10\angle 0^\circ + 1\angle 120^\circ \cdot 0 + 1\angle 240^\circ \cdot 10\angle + 120^\circ \\ 1 \cdot 10\angle 0^\circ + 1\angle 240^\circ \cdot 0 + 1\angle 120^\circ \cdot 10\angle + 120^\circ \\ 1 \cdot 10\angle 0^\circ + 1 \cdot 0 + 1 \cdot 10\angle + 120^\circ \end{bmatrix} = & (8.19) \\ &= \begin{bmatrix} 6.667\angle 0^\circ \\ 3.333\angle - 60^\circ \\ 3.333\angle 60^\circ \end{bmatrix} \\ \overline{I}_n &= \overline{I}_a + \overline{I}_b + \overline{I}_c = 10\angle 0^\circ + 0 + 10\angle + 120^\circ = 10\angle 60^\circ = 3\overline{I}_{-0} & (8.20) \end{aligned}$$

As given in the example, the current through the neutral-ground conductor is three times as large as the zero-sequence current.

8.1.1 Power calculations under unbalanced conditions

Based on the voltage and current phasors of each phase, the three-phase complex power can be calculated as

$$\overline{S} = P + jQ = \overline{U}_a \overline{I}_a^* + \overline{U}_b \overline{I}_b^* + \overline{U}_c \overline{I}_c^* = \mathbf{U_{Ph}^t I_{Ph}^*} \tag{8.21}$$

By introducing symmetrical components, the expression above can be converted to

$$\overline{S} = \mathbf{U_{Ph}^t I_{Ph}^*} = (\mathbf{TU_s})^\mathbf{t} (\mathbf{TI_s})^* = \mathbf{U_s^t T^t T^* I_s^*} \tag{8.22}$$

where t stands for transpose. The expression $\mathbf{T^t T^*}$ can be written as

$$\mathbf{T^t T^*} = \begin{bmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.23}$$

i.e. the transformation is not power invariant since $\mathbf{U_{Ph}^t I_{Ph}^*} \neq \mathbf{U_s^t I_s^*}$, see also Appendix B. Equation (8.22) can be rewritten as

$$\overline{S} = 3\,\mathbf{U_s^t}\,\mathbf{I_s^*} = 3\,\overline{U}_{-1}\,\overline{I}_{-1}^* + 3\,\overline{U}_{-2}\,\overline{I}_{-2}^* + 3\,\overline{U}_{-0}\,\overline{I}_{-0}^* \tag{8.24}$$

Since the magnitude of the line-to-line voltages are $\sqrt{3}$ times the line-to-neutral voltages and $S_{base} = \sqrt{3} \cdot U_{base} \cdot I_{base}$, the introduction of the per-unit system gives that equation (8.24) can be rewritten as

$$\overline{S}_{pu} = \frac{\sqrt{3}(\sqrt{3}\mathbf{U_s})^t \cdot \mathbf{I_s^*}}{\sqrt{3} \cdot U_{base} \cdot I_{base}} = \overline{U}_{pu-1}\,\overline{I}^*_{pu-1} + \overline{U}_{pu-2}\,\overline{I}^*_{pu-2} + \overline{U}_{pu-0}\,\overline{I}^*_{pu-0} \quad \text{pu} \tag{8.25}$$

This implies that the total power (in per-unit value) in an unbalanced system can be expressed by the sum of the symmetrical components of power. The total power in physical unit can be obtained by multiplying $\overline{S}_{pu}$ with S_{base}, i.e.

$$\overline{S} = \overline{S}_{pu}\,S_{base} = \overline{U}_{pu-1}\,\overline{I}^*_{pu-1}\,S_{base} + \overline{U}_{pu-2}\,\overline{I}^*_{pu-2}\,S_{base} + \overline{U}_{pu-0}\,\overline{I}^*_{pu-0}\,S_{base} \quad \text{MVA} \tag{8.26}$$

8.2 Sequence circuits of power system components

8.2.1 Transformers

In the analysis of three-phase circuits under unbalanced conditions, the transformer is represented by its positive-, negative- and zero-sequence impedances. These can be determined by analyzing the three-phase transformer, e.g. the Y0-Δ connected shown in Figure 8.2.

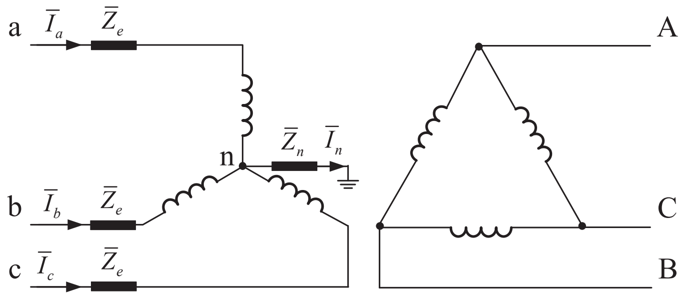


Figure 8.2. Y0-Δ connected transformer with neutral point grounded through an impedance $\overline{Z}_n$.

The impedance $\overline{Z}_e$ represents the equivalent impedance of each phase and consists of both leakage reactance of the primary and secondary windings as well as the resistance of the windings, i.e. the windings shown in the figure are considered as ideal windings. The magnetizing current of the transformer can be neglected, i.e. the magnetizing impedance is assumed to be infinitely large.

By using the direction of currents as shown in Figure 8.2, the following expressions for the three phases of the transformer can be held

$$\begin{aligned}\Delta\overline{U}_a &= \overline{I}_a\overline{Z}_e + \overline{I}_n\overline{Z}_n \\ \Delta\overline{U}_b &= \overline{I}_b\overline{Z}_e + \overline{I}_n\overline{Z}_n \\ \Delta\overline{U}_c &= \overline{I}_c\overline{Z}_e + \overline{I}_n\overline{Z}_n\end{aligned} \tag{8.27}$$

Since $\overline{I}_n = \overline{I}_a + \overline{I}_b + \overline{I}_c$, this can be rewritten as

$$\begin{aligned} \Delta\overline{U}_a &= \overline{I}_a\left(\overline{Z}_e + \overline{Z}_n\right) + \overline{I}_b\overline{Z}_n + \overline{I}_c\overline{Z}_n \\ \Delta\overline{U}_b &= \overline{I}_a\overline{Z}_n + \overline{I}_b\left(\overline{Z}_e + \overline{Z}_n\right) + \overline{I}_c\overline{Z}_n \\ \Delta\overline{U}_c &= \overline{I}_a\overline{Z}_n + \overline{I}_b\overline{Z}_n + \overline{I}_c\left(\overline{Z}_e + \overline{Z}_n\right) \end{aligned} \tag{8.28}$$

which can be written on matrix form

$$\mathbf{\Delta U_{Ph}} = \begin{bmatrix} \Delta\overline{U}_a \\ \Delta\overline{U}_b \\ \Delta\overline{U}_c \end{bmatrix} = \begin{bmatrix} \overline{Z}_e + \overline{Z}_n & \overline{Z}_n & \overline{Z}_n \\ \overline{Z}_n & \overline{Z}_e + \overline{Z}_n & \overline{Z}_n \\ \overline{Z}_n & \overline{Z}_n & \overline{Z}_e + \overline{Z}_n \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} = \mathbf{Z_{tr}I_{Ph}} \tag{8.29}$$

Transforming the above phase quantities to the symmetrical components, the following can be obtained.

$$\mathbf{Z_{trs}} = \mathbf{T^{-1}Z_{tr}T} = \begin{bmatrix} \overline{Z}_e & 0 & 0 \\ 0 & \overline{Z}_e & 0 \\ 0 & 0 & \overline{Z}_e + 3\overline{Z}_n \end{bmatrix} \tag{8.30}$$

i.e.

$$\begin{aligned} \overline{Z}_{t-1} &= \overline{Z}_e = \text{ positive-sequence impedance} \\ \overline{Z}_{t-2} &= \overline{Z}_e = \text{ negative-sequence impedance} \\ \overline{Z}_{t-0} &= \overline{Z}_e + 3\overline{Z}_n = \text{ zero-sequence impedance} \end{aligned} \tag{8.31}$$

As given above, the positive- and negative-sequence impedances are the same and equal to $\overline{Z}_e$. That $\overline{Z}_{t-1} = \overline{Z}_{t-2}$ is not surprising since the transformer impedance does not change if the phase ordering is changed from *abc* (positive-sequence) to *acb* (negative-sequence).

The zero-sequence impedance includes $\overline{Z}_e$ but a factor of $3\overline{Z}_n$ is added where $\overline{Z}_n$ is the impedance connected between the transformer neutral and the ground. If $\overline{Z}_n = 0$, the zero-sequence impedance will be equal to $\overline{Z}_e$. Note that to obtain zero-sequence currents, it must be a connection between the transformer neutral and the ground.

Whereas the positive- and negative-sequence impedances of the transformer are independent on from which side of the transformer the analysis is performed, the zero-sequence impedance can vary with a large amount. Figure 8.3 a) shows a Y0-Y0 connected transformer through which zero-sequence currents can flow since both sides are grounded. The zero-sequence impedance is given by $\overline{Z}_{t-0} = \overline{Z}_e + 3(\overline{Z}_n + \overline{Z}_N)$.

Figure 8.3 b)-c) show a Y-Y connected and a Y0-Y connected transformers through which zero-sequence currents cannot flow, since no zero-sequence currents can flow in the primary winding unless a zero-sequence currents flow in the secondary winding.

Figure 8.3 d) shows a Y0-Δ connected transformer through which zero-sequence currents can flow, but only on the Y0-side since a neutral-ground conductor exists. Note that due to the induced currents and an mmf-balance, there exist circulating currents in the Δ-winding, but they cannot flow into a system connected to the Δ-winding, i.e. $\bar{I}_{A-0} = \bar{I}_{B-0} = \bar{I}_{C-0} = 0$.

In Figure 8.3 e)–f), no zero-sequence currents can flow on either side due to the connection types.

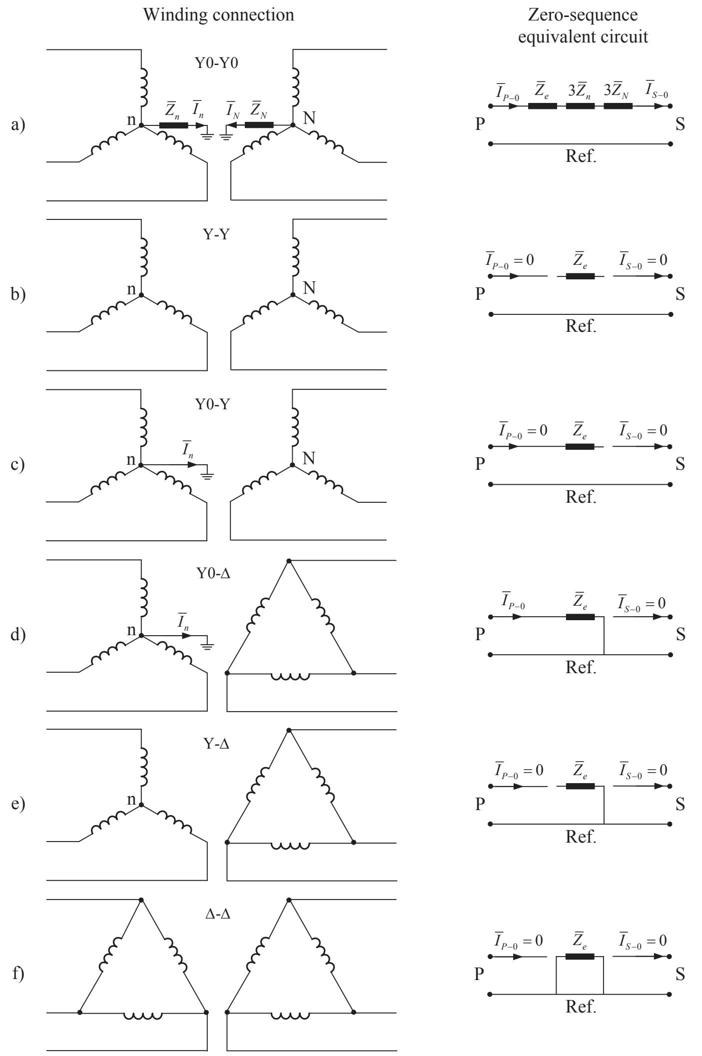


Figure 8.3. Zero-sequence equivalent circuits of transformers with different winding connections.

8.2.2 Impedance loads

A three-phase impedance load is normally Y- or Δ-connected as shown in Figure 8.4.

The neutral of a Y-connected load may be grounded with or without an impedance. Then it is termed Y0-connected. For the $Y0$-connected load shown in Figure 8.4 a), we have:

$$\mathbf{U_{Ph}} = \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \begin{bmatrix} \overline{Z}_a + \overline{Z}_n & \overline{Z}_n & \overline{Z}_n \\ \overline{Z}_n & \overline{Z}_b + \overline{Z}_n & \overline{Z}_n \\ \overline{Z}_n & \overline{Z}_n & \overline{Z}_c + \overline{Z}_n \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} = \mathbf{Z_{LDPh} I_{Ph}} \tag{8.32}$$

This equation can be transformed to symmetrical components as follows:

$$\begin{aligned} \mathbf{U_{Ph}} &= \mathbf{T U_s} = \mathbf{Z_{LDPh} I_{Ph}} = \mathbf{Z_{LDPh} T\, I_s} \\ &\Rightarrow \\ \mathbf{U_s} &= \mathbf{T^{-1}\, Z_{LDPh} T\, I_s} = \mathbf{Z_{LDs} I_s} \end{aligned} \tag{8.33}$$

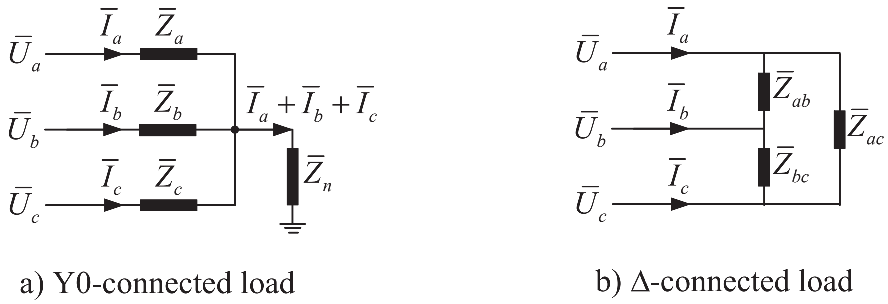


Figure 8.4. Two possible load configurations

where

$$\begin{aligned}
\mathbf{Z_{LDs}} &\equiv \mathbf{T^{-1}}\,\mathbf{Z_{LDPh}}\mathbf{T} = \qquad (8.34)\\
&= \frac{1}{3}\begin{bmatrix} \overline{Z}_a+\overline{Z}_b+\overline{Z}_c & \overline{Z}_a+\alpha^2\overline{Z}_b+\alpha\overline{Z}_c & \overline{Z}_a+\alpha\overline{Z}_b+\alpha^2\overline{Z}_c \\ \overline{Z}_a+\alpha\overline{Z}_b+\alpha^2\overline{Z}_c & \overline{Z}_a+\overline{Z}_b+\overline{Z}_c & \overline{Z}_a+\alpha^2\overline{Z}_b+\alpha\overline{Z}_c \\ \overline{Z}_a+\alpha^2\overline{Z}_b+\alpha\overline{Z}_c & \overline{Z}_a+\alpha\overline{Z}_b+\alpha^2\overline{Z}_c & \overline{Z}_a+\overline{Z}_b+\overline{Z}_c+9\overline{Z}_n \end{bmatrix}
\end{aligned}$$

As indicated in this matrix, there are non-diagonal elements that are nonzero, i.e. there exists couplings between the positive-, negative- and zero-sequences. A special case is when $\overline{Z}_a = \overline{Z}_b = \overline{Z}_c$. In this special case, $\mathbf{Z_{LDs}}$ can be written as

$$\mathbf{Z_{LDs}} = \begin{bmatrix} \overline{Z}_a & 0 & 0 \\ 0 & \overline{Z}_a & 0 \\ 0 & 0 & \overline{Z}_a+3\overline{Z}_n \end{bmatrix} \qquad (8.35)$$

For a symmetric $Y0$-connected load $\overline{Z}_{LD-1} = \overline{Z}_{LD-2} = \overline{Z}_a$ and $\overline{Z}_{LD-0} = \overline{Z}_a + 3\overline{Z}_n$. If $\overline{Z}_n = 0$ then $\overline{Z}_{LD-1} = \overline{Z}_{LD-2} = \overline{Z}_{LD-0}$. However, if the neutral of the load is not grounded, i.e a Y-connected load, then $\overline{Z}_n = \infty = \overline{Z}_{LD-0}$ which means that no zero-sequence currents can flow.

For the Δ-connected load shown in Figure 8.4 b), the impedance can be Δ-Y transformed which results in a Y-connected load:

$$\overline{Z}_a = \frac{\overline{Z}_{ab}\overline{Z}_{ac}}{\overline{Z}_{ab}+\overline{Z}_{ac}+\overline{Z}_{bc}} \qquad (8.36)$$

$$\overline{Z}_b = \frac{\overline{Z}_{ab}\overline{Z}_{bc}}{\overline{Z}_{ab}+\overline{Z}_{ac}+\overline{Z}_{bc}} \qquad (8.37)$$

$$\overline{Z}_c = \frac{\overline{Z}_{ac}\overline{Z}_{bc}}{\overline{Z}_{ab}+\overline{Z}_{ac}+\overline{Z}_{bc}} \qquad (8.38)$$

$$\overline{Z}_n = \infty \qquad (8.39)$$

For a symmetric Δ-connected load, i.e. $\overline{Z}_{ab} = \overline{Z}_{bc} = \overline{Z}_{ac}$, the symmetrical components can be calculated by using equations (8.35) to (8.39) :

$$\overline{Z}_{LD-1} = \overline{Z}_{ab}/3 \qquad (8.40)$$

$$\overline{Z}_{LD-2} = \overline{Z}_{ab}/3 \qquad (8.41)$$

$$\overline{Z}_{LD-0} = \infty \qquad (8.42)$$

8.2.3 Transmission line

Series impedance of single-phase overhead line

The theory of having an overhead line using the ground as a return conductor was discussed by Carson in 1923. Carson considered a single conductor of unity length (e.g. one meter) in parallel with the ground, see Figure 8.5.

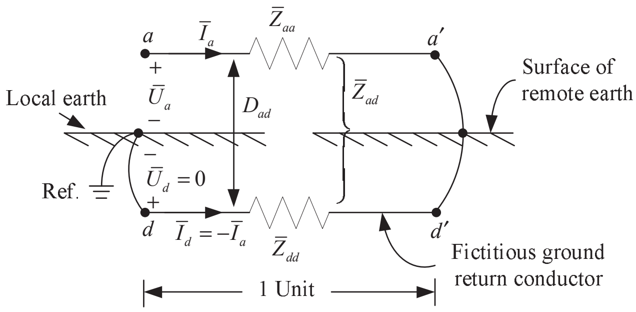


Figure 8.5. Carson's single-phase overhead line using the ground as return path

The current $\overline{I}_a$ flows in the conductor using the ground between $d - d'$ as return path. The ground is assumed to have an uniform resistance and an infinite extension. The current $\overline{I}_d$ ($=-\overline{I}_a$) is distributed over a large area, flowing along the ways of least resistance. Kirchhoff's law about the same voltage drop along each path is fulfilled. It has been shown that these distributed return paths may, in the analysis, be replaced by a single return conductor having a radius ε_d located at a distance D_{ad} from the overhead line according to Figure 8.5. The distance D_{ad} is a function of the resistivity of the ground ρ. The distance D_{ad} increases as the resistivity ρ increases.

The inductance of this circuit can be calculated as

$$L_a = \underbrace{\frac{\mu}{2\pi} \ln \frac{1}{D_a}}_{L_{aa}} + \underbrace{\frac{\mu}{2\pi} \ln \frac{1}{D_d}}_{L_{dd}} - 2 \underbrace{\frac{\mu}{2\pi} \ln \frac{1}{D_{ad}}}_{L_{ad}} = \frac{\mu}{2\pi} \left(\ln \frac{D_{ad}}{D_a} + \ln \frac{D_{ad}}{D_d} \right) \tag{8.43}$$

where

$$\begin{aligned} \mu &= \text{ the permeability of the conductor} \\ D_a &= e^{-1/4}\varepsilon_a \text{ for a single conductor with radius } \varepsilon_a \\ D_d &= e^{-1/4}\varepsilon_d \text{ for a return conductor in ground with radius } \varepsilon_d \end{aligned}$$

The inductance can according to equation (8.43) be divided into three parts, two *apparent self inductances* (L_{aa}, L_{dd}) and one *apparent mutual inductance* (L_{ad}). Note that these quantities are only mathematical quantities without any physical meaning. For instance, they do not have correct unit inside the ln-sign. It is only after the summation they achieve a physical meaning. Hopefully, the different part expressions will simplify the understanding of the behavior of a *three-phase* line. The total series reactance of this single-phase conductor is

$$X_a = \omega L_a = \omega(L_{aa} + L_{dd} - 2L_{ad}) \tag{8.44}$$

By using this line model, having apparent inductances, the voltage drop for a single-phase line can be calculated as

$$\begin{bmatrix} \overline{U}_{aa'} \\ \overline{U}_{dd'} \end{bmatrix} = \begin{bmatrix} \overline{U}_a - \overline{U}_{a'} \\ \overline{U}_d - \overline{U}_{d'} \end{bmatrix} = \begin{bmatrix} \overline{z}_{aa} & \overline{z}_{ad} \\ \overline{z}_{ad} & \overline{z}_{dd} \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ -\overline{I}_a \end{bmatrix} \text{ V/length unit} \tag{8.45}$$

where $\overline{U}_a, \overline{U}_{a'}, \overline{U}_d$ and $\overline{U}_{d'}$ are given in proportion to the same reference. Since $\overline{U}_d = 0$ and $\overline{U}_{a'} - \overline{U}_{d'} = 0$, $\overline{U}_a$ can be obtained by subtracting the two equations from each other :

$$\overline{U}_a = (\overline{z}_{aa} + \overline{z}_{dd} - 2\overline{z}_{ad})\overline{I}_a = \overline{Z}_a I_a \tag{8.46}$$

By definition

$$\overline{Z}_a \equiv \overline{z}_{aa} + \overline{z}_{dd} - 2\overline{z}_{ad} \quad \Omega\text{/length unit} \tag{8.47}$$

The impedances in this equation can be calculated as

$$\begin{aligned} \overline{z}_{aa} &= r_a + jx_{aa} = r_a + j\omega L_{aa} \quad \Omega\text{/length unit} \\ \overline{z}_{dd} &= r_d + jx_{dd} = r_d + j\omega L_{dd} \quad \Omega\text{/length unit} \\ \overline{z}_{ad} &= jx_{ad} = j\omega L_{ad} \quad \Omega\text{/length unit} \\ \overline{Z}_a &= r_a + r_d + jX_a \quad \Omega\text{/length unit} \end{aligned} \tag{8.48}$$

where

$$\begin{aligned} r_a &= \text{conductor resistance per length unit} \\ r_d &= \text{ground resistance per length unit} \end{aligned}$$

Series impedance of a three-phase overhead line

In order to obtain the series impedance of a three-phase line, the calculations are performed in the same way as for the single-phase line. In Figure 8.6, the impedances, voltages and currents of the line are given.

Since all conductors are grounded at a', b', c', the following are valid

$$\begin{aligned} \overline{U}_{a'} - \overline{U}_{d'} = 0 \quad , \quad \overline{U}_{b'} - \overline{U}_{d'} = 0 \quad , \quad \overline{U}_{c'} - \overline{U}_{d'} = 0 \\ \overline{I}_d = -(\overline{I}_a + \overline{I}_b + \overline{I}_c) \end{aligned} \tag{8.49}$$

The voltage drop over the conductors can be calculated as

$$\begin{bmatrix} \overline{U}_{aa'} \\ \overline{U}_{bb'} \\ \overline{U}_{cc'} \\ \overline{U}_{dd'} \end{bmatrix} = \begin{bmatrix} \overline{U}_a - \overline{U}_{a'} \\ \overline{U}_b - \overline{U}_{b'} \\ \overline{U}_c - \overline{U}_{c'} \\ \overline{U}_d - \overline{U}_{d'} \end{bmatrix} = \begin{bmatrix} \overline{z}_{aa} & \overline{z}_{ab} & \overline{z}_{ac} & \overline{z}_{ad} \\ \overline{z}_{ab} & \overline{z}_{bb} & \overline{z}_{bc} & \overline{z}_{bd} \\ \overline{z}_{ac} & \overline{z}_{bc} & \overline{z}_{cc} & \overline{z}_{cd} \\ \overline{z}_{ad} & \overline{z}_{bd} & \overline{z}_{cd} & \overline{z}_{dd} \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \\ \overline{I}_d \end{bmatrix} \text{ V/length unit} \tag{8.50}$$

In a similar way as for the single-phase conductor, the impedances in equation (8.50) are apparent without any physical relevance. With $\overline{U}_d = 0$ and by using equation (8.49), the fourth row can be subtracted from the first row in equation (8.50) which gives

$$\begin{aligned} \overline{U}_a - (\overline{U}_{a'} - \overline{U}_{d'}) &= (\overline{z}_{aa} - 2\overline{z}_{ad} + \overline{z}_{dd})\overline{I}_a + (\overline{z}_{ab} - \overline{z}_{ad} - \overline{z}_{bd} + \overline{z}_{dd})\overline{I}_b + \\ &+ (\overline{z}_{ac} - \overline{z}_{ad} - \overline{z}_{cd} + \overline{z}_{dd})\overline{I}_c \end{aligned} \tag{8.51}$$

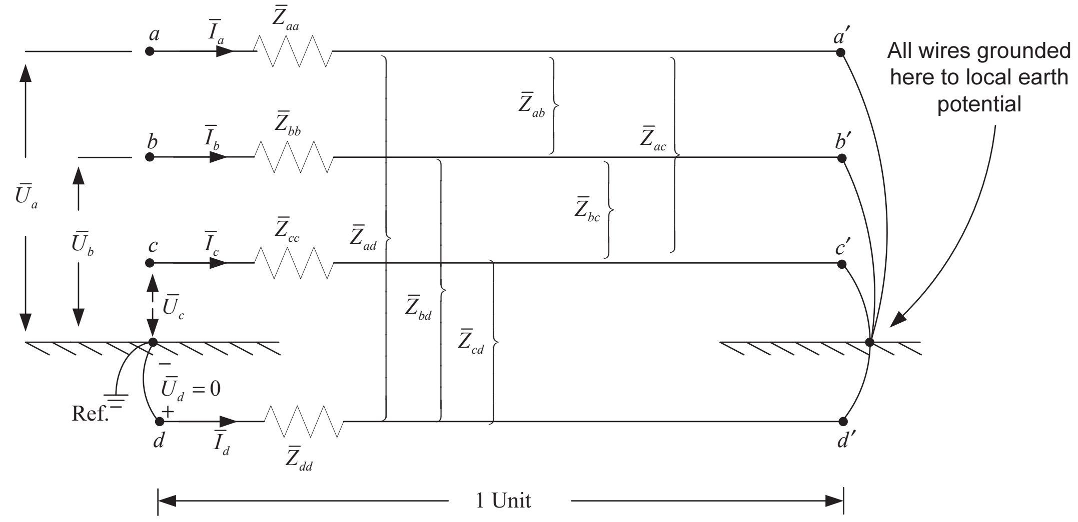


Figure 8.6. Three-phase overhead line with ground as return path

This can be simplified to $\overline{U}_a = \overline{Z}_{aa}\overline{I}_a + \overline{Z}_{ab}\overline{I}_b + \overline{Z}_{ac}\overline{I}_c$. The impedances $\overline{Z}_{aa}, \overline{Z}_{ab}$ and $\overline{Z}_{ac}$ are defined below. Note that when $\overline{I}_b = \overline{I}_c = 0$, the impedance $\overline{Z}_{aa}$ is exactly the impedance of a single-phase line using the ground as return path as described in section 8.2.3. If the calculations above are repeated for the phases b and c, the following can be obtained

$$\begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ac} \\ \overline{Z}_{ab} & \overline{Z}_{bb} & \overline{Z}_{bc} \\ \overline{Z}_{ac} & \overline{Z}_{bc} & \overline{Z}_{cc} \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} \text{ V/length unit} \tag{8.52}$$

where

$$\begin{aligned} \overline{Z}_{aa} &= \overline{z}_{aa} - 2\overline{z}_{ad} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \\ \overline{Z}_{bb} &= \overline{z}_{bb} - 2\overline{z}_{bd} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \\ \overline{Z}_{cc} &= \overline{z}_{cc} - 2\overline{z}_{cd} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \\ \overline{Z}_{ab} = \overline{Z}_{ba} &= \overline{z}_{ab} - \overline{z}_{ad} - \overline{z}_{bd} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \\ \overline{Z}_{bc} = \overline{Z}_{cb} &= \overline{z}_{bc} - \overline{z}_{bd} - \overline{z}_{cd} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \\ \overline{Z}_{ac} = \overline{Z}_{ca} &= \overline{z}_{ac} - \overline{z}_{ad} - \overline{z}_{cd} + \overline{z}_{dd} \;\; \Omega/\text{length unit} \end{aligned} \tag{8.53}$$

The impedances can be calculated in a similar way as shown in equations (8.43) and (8.48). It is important to concern the coupling between the phases. A current flowing in one phase will influence the voltage drop in other phases. The replacing of a three-phase line with three parallel impedances, is an approximation which gives that all non-diagonal element of the Z-bus matrix in equation (8.52) are neglected. In other words, the mutual inductance between the conductors are neglected. The error this simplification gives is dependent on several things, e.g. the distance between the conductors, the length of the conductors and the magnitude of the currents in the conductors.

Symmetrical components of the series impedance of a three-phase line

Symmetrical components are often used in the analysis of power systems having three-phase lines, in order to simplify the complicated cross-couplings that exist between the phases. The quantities in equation (8.52) can be defined as :

$$\begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \mathbf{U_{Ph}} = \mathbf{Z_{Ph}} \mathbf{I_{Ph}} = \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ac} \\ \overline{Z}_{ab} & \overline{Z}_{bb} & \overline{Z}_{bc} \\ \overline{Z}_{ac} & \overline{Z}_{bc} & \overline{Z}_{cc} \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} \tag{8.54}$$

The voltage vector ($\mathbf{U_{Ph}}$) and current vector ($\mathbf{I_{Ph}}$) can be replaced by the corresponding symmetrical component multiplied with matrix $\mathbf{T}$ according to the section on symmetrical components :

$$\mathbf{U_{Ph}} = \mathbf{T}\mathbf{U_s} = \mathbf{Z_{Ph}}\mathbf{T}\mathbf{I_s} = \mathbf{Z_{Ph}}\mathbf{I_{Ph}} \tag{8.55}$$

This equation can be rewritten as

$$\mathbf{U_s} = \mathbf{T^{-1}}\mathbf{Z_{Ph}}\mathbf{T}\mathbf{I_s} = \mathbf{Z_s}\mathbf{I_s} \tag{8.56}$$

If a symmetrical overhead line (or cable) is assumed, i.e. $\overline{Z}_{aa} = \overline{Z}_{bb} = \overline{Z}_{cc}$ and $\overline{Z}_{ab} = \overline{Z}_{bc} = \overline{Z}_{ac}$, the following is obtained

$$\begin{aligned} \mathbf{Z_s} = \mathbf{T^{-1}}\mathbf{Z_{Ph}}\mathbf{T} &= \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ac} \\ \overline{Z}_{ab} & \overline{Z}_{bb} & \overline{Z}_{bc} \\ \overline{Z}_{ac} & \overline{Z}_{bc} & \overline{Z}_{cc} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} \overline{Z}_{aa} - \overline{Z}_{ab} & 0 & 0 \\ 0 & \overline{Z}_{aa} - \overline{Z}_{ab} & 0 \\ 0 & 0 & \overline{Z}_{aa} + 2\overline{Z}_{ab} \end{bmatrix} \end{aligned} \tag{8.57}$$

Equation (8.56) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} \overline{U}_{-1} \\ \overline{U}_{-2} \\ \overline{U}_{-0} \end{bmatrix} = \mathbf{U_s} = \mathbf{Z_s}\,\mathbf{I_s} = \\ &= \begin{bmatrix} \overline{Z}_{aa} - \overline{Z}_{ab} & 0 & 0 \\ 0 & \overline{Z}_{aa} - \overline{Z}_{ab} & 0 \\ 0 & 0 & \overline{Z}_{aa} + 2\overline{Z}_{ab} \end{bmatrix} \begin{bmatrix} \overline{I}_{-1} \\ \overline{I}_{-2} \\ \overline{I}_{-0} \end{bmatrix} = \begin{bmatrix} \overline{Z}_{-1}\overline{I}_{-1} \\ \overline{Z}_{-2}\overline{I}_{-2} \\ \overline{Z}_{-0}\overline{I}_{-0} \end{bmatrix} \end{aligned} \tag{8.58}$$

where

$$\begin{aligned} \overline{Z}_{-1} &= \overline{Z}_{aa} - \overline{Z}_{ab} = \textit{positive-sequence impedance} \\ \overline{Z}_{-2} &= \overline{Z}_{aa} - \overline{Z}_{ab} = \textit{negative-sequence impedance} \\ \overline{Z}_{-0} &= \overline{Z}_{aa} + 2\overline{Z}_{ab} = \textit{zero-sequence impedance} \end{aligned} \tag{8.59}$$

By inserting the expressions used in equation (8.53) into equation (8.59), the following can be obtained

$$\begin{aligned} \overline{Z}_{-1} = \overline{Z}_{-2} &= \overline{z}_{aa} - \overline{z}_{ab} \\ \overline{Z}_{-0} &= \overline{z}_{aa} + 2\overline{z}_{ab} - 6\overline{z}_{ad} + 3\overline{z}_{dd} \end{aligned} \tag{8.60}$$

Note that the coupling to ground are not present in the expressions for the positive- and negative-sequence impedances, i.e. the elements having index d in the Z-bus matrix in equation (8.50) are not included. This means that the zero-sequence current is zero in the positive- and negative-sequence reference frame, which is quite logical. All couplings to ground are represented in the zero-sequence impedance. As indicated above, a line by using this model, can be represented as three non-coupled components : positive-, negative-, and zero-sequence components. It should be pointed out that some loss of information will occur when using this model. For example, if only positive-, negative-, and zero-sequence data are given, the potential of the ground, $\overline{U}_{d'}$ in Figure 8.6, cannot be calculated. To calculate that potential, more detailed data are needed. The line model introduced in subsection 6.1.2, is based on positive-sequence data only, since symmetrical conditions are assumed.

Equivalent diagram of the series impedance of a line

As given above, for a symmetrical line $\overline{Z}_{-1} = \overline{Z}_{-2}$. Assume that this line can be replaced by an equivalent circuit according to Figure 8.7, i.e. three phase impedances $\overline{Z}_\alpha$ and one return

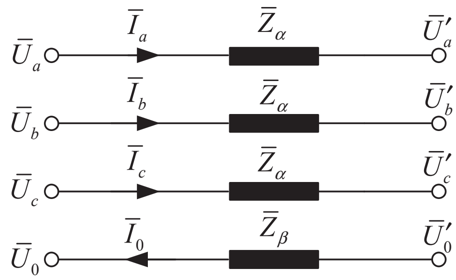


Figure 8.7. Equivalent diagram of the series impedance of a line

impedance $\overline{Z}_\beta$ where the mutual inductance between the phases is assumed zero. With three phases and one return path, as given by the equivalent in Figure 8.7, the following is valid

$$\overline{I}_0 = \overline{I}_a + \overline{I}_b + \overline{I}_c \tag{8.61}$$

By using equation (8.61), the voltage drop between the phases and the return conductor can be calculated as

$$\begin{aligned}
\overline{U}'_a - \overline{U}'_0 &= \overline{U}_a - \overline{U}_0 - \overline{I}_a \cdot \overline{Z}_\alpha - (\overline{I}_a + \overline{I}_b + \overline{I}_c)\overline{Z}_\beta \\
\overline{U}'_b - \overline{U}'_0 &= \overline{U}_b - \overline{U}_0 - \overline{I}_b \cdot \overline{Z}_\alpha - (\overline{I}_a + \overline{I}_b + \overline{I}_c)\overline{Z}_\beta \\
\overline{U}'_c - \overline{U}'_0 &= \overline{U}_c - \overline{U}_0 - \overline{I}_c \cdot \overline{Z}_\alpha - (\overline{I}_a + \overline{I}_b + \overline{I}_c)\overline{Z}_\beta
\end{aligned} \tag{8.62}$$

which can be rewritten to matrix form

$$\begin{bmatrix} \overline{U}'_a - \overline{U}'_0 \\ \overline{U}'_b - \overline{U}'_0 \\ \overline{U}'_c - \overline{U}'_0 \end{bmatrix} = \begin{bmatrix} \overline{U}_a - \overline{U}_0 \\ \overline{U}_b - \overline{U}_0 \\ \overline{U}_c - \overline{U}_0 \end{bmatrix} - \begin{bmatrix} \overline{Z}_\alpha + \overline{Z}_\beta & \overline{Z}_\beta & \overline{Z}_\beta \\ \overline{Z}_\beta & \overline{Z}_\alpha + \overline{Z}_\beta & \overline{Z}_\beta \\ \overline{Z}_\beta & \overline{Z}_\beta & \overline{Z}_\alpha + \overline{Z}_\beta \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} \tag{8.63}$$

or

$$\mathbf{U}'_{\mathbf{Ph}} = \mathbf{U}_{\mathbf{Ph}} - \mathbf{Z}_{\alpha\beta}\mathbf{I}_{\mathbf{Ph}} \tag{8.64}$$

Since the matrix $\mathbf{Z}_{\alpha\beta}$ is both symmetric and cyclo-symmetric, it can represent a line according to the assumption made. The matrix $\mathbf{Z}_{\alpha\beta}$ can be converted to symmetrical components by using equation (8.57) :

$$\mathbf{Z}_{\mathbf{s}\alpha\beta} = \mathbf{T}^{-1}\mathbf{Z}_{\alpha\beta}\mathbf{T} = \begin{bmatrix} \overline{Z}_\alpha & 0 & 0 \\ 0 & \overline{Z}_\alpha & 0 \\ 0 & 0 & \overline{Z}_\alpha + 3\overline{Z}_\beta \end{bmatrix} \tag{8.65}$$

When the symmetrical components $\overline{Z}_1 = \overline{Z}_2$ and $\overline{Z}_0$ for the line are known, the following is obtained

$$\begin{aligned} \overline{Z}_\alpha &= \overline{Z}_{-1} \\ \overline{Z}_\beta &= \tfrac{\overline{Z}_{-0} - \overline{Z}_{-1}}{3} \end{aligned} \tag{8.66}$$

With these values of $\overline{Z}_\alpha$ and $\overline{Z}_\beta$, the equivalent in Figure 8.7 can be used, together with equation (8.63), to calculate the voltage drop between the phases and the return conductor $(= \mathbf{U}_{\mathbf{Ph}} - \mathbf{U}'_{\mathbf{Ph}})$ as a function of the phase currents $(= \mathbf{I}_{\mathbf{Ph}})$.

Note that the equivalent **cannot** be used to calculate e.g. $\overline{U}'_0 - \overline{U}_0$ or $\overline{U}'_a - \overline{U}_a$ but only e.g. $(\overline{U}'_a - \overline{U}'_0) - (\overline{U}_a - \overline{U}_0)$.

Example 8.3 *Solve Example 2.5 by using symmetrical components.*

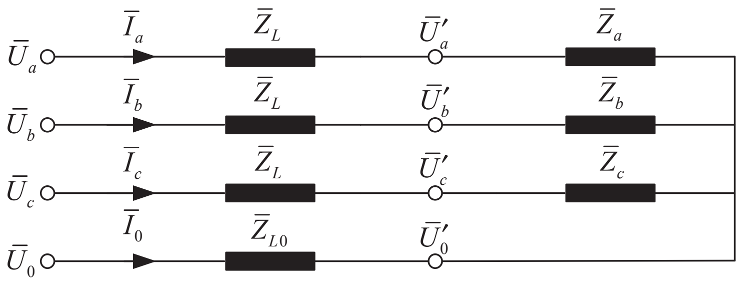


Figure 8.8. Network diagram of the system in Example 8.3

Solution

According to the solutions in Example 2.5, the impedances of interest are $\overline{Z}_L = 2.3 + j0.16$ Ω, $\overline{Z}_{L0} = 2.3 + j0.03\ \Omega$, $\overline{Z}_a = 47.9 + j4.81\ \Omega$, $\overline{Z}_b = 15.97 + j1.60\ \Omega$, $\overline{Z}_c = 23.96{+}j2.40\ \Omega$.

The symmetrical components of the line will first be calculated. Note that the line in the example is given in the same way as the equivalent. The symmetrical components can be calculated by using equation (8.65) :

$$\begin{aligned} \overline{Z}_{-1} = \overline{Z}_{-2} &= \overline{Z}_L = 2.3 + j0.16\ \Omega \\ \overline{Z}_{-0} &= \overline{Z}_L + 3\overline{Z}_{L0} = 9.2 + j0.25\ \Omega \end{aligned} \tag{8.67}$$

which gives that

$$\mathbf{Z_s} = \begin{bmatrix} \overline{Z}_{-1} & 0 & 0 \\ 0 & \overline{Z}_{-2} & 0 \\ 0 & 0 & \overline{Z}_{-0} \end{bmatrix} = \begin{bmatrix} 2.3 + j0.16 & 0 & 0 \\ 0 & 2.3 + j0.16 & 0 \\ 0 & 0 & 9.2 + j0.25 \end{bmatrix} \tag{8.68}$$

The symmetrical components for the load can be calculated by using equation (8.57)

$$\begin{aligned} \mathbf{Z_{LDs}} &= \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \overline{Z}_a & 0 & 0 \\ 0 & \overline{Z}_b & 0 \\ 0 & 0 & \overline{Z}_c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 29.28 + j2.94 & 9.09 + j3.24 & 9.55 - j1.37 \\ 9.55 - j1.37 & 29.28 + j2.94 & 9.09 + j3.24 \\ 9.09 + j3.24 & 9.55 - j1.37 & 29.28 + j2.94 \end{bmatrix} \Omega \end{aligned} \tag{8.69}$$

The applied voltage is symmetric, i.e. it has only one sequence, the positive one :

$$\mathbf{U_s} = \mathbf{T^{-1}U_{Ph}} = \begin{bmatrix} 220\angle 0^\circ \\ 0 \\ 0 \end{bmatrix} \text{ V} \tag{8.70}$$

The equation for this un-symmetric three-phase network can be described as

$$\mathbf{U_s} = (\mathbf{Z_s} + \mathbf{Z_{LDs}})\mathbf{I_s} \tag{8.71}$$

which can be rewritten as

$$\mathbf{I_s} = (\mathbf{Z_s} + \mathbf{Z_{LDs}})^{-1}\mathbf{U_s} = \begin{bmatrix} 8.11\angle -5.51^\circ \\ 2.22\angle 149.09^\circ \\ 1.75\angle -155.89^\circ \end{bmatrix} \text{ A} \tag{8.72}$$

The symmetrical components for the voltage at the load can be calculated as

$$\mathbf{U_{LDs}} = \mathbf{Z_{LDs}I_s} = \begin{bmatrix} 201.32\angle -0.14^\circ \\ 5.13\angle -26.93^\circ \\ 16.10\angle 25.67^\circ \end{bmatrix} \text{ V} \tag{8.73}$$

The power obtained in the radiators can be calculated by using equation (8.24)

$$\overline{S} = 3\mathbf{U^t_{LDs}I^*_s} = 4754 + j477 \text{ VA} \tag{8.74}$$

i.e. the thermal power is 4754 W.

As given above, only the voltage drop at the load and the load currents can be calculated by using the symmetrical components. The ground potential at the load cannot be calculated, but that is usually of no interest.

Previously, in Example 2.5, 4.1 and 4.2, the ground potential at the load has been calculated by using other types of circuit analyses. It should be pointed out that the value of the ground potential has no physical interpretation if the value of $\overline{Z}_L$ and $\overline{Z}_{L0}$ has been obtained by using the symmetrical components of the line according to equation (8.66). As given by the solutions, the load demand, phase voltages at the load and the currents at the load are physically correct by using either one of the four methods of solution.

Shunt capacitance of a three-phase line

The line resistance and inductance are components that together form the *series* impedance of the line. The capacitance that is of interest in this section, forms the *shunt* component.

The series component, usually the inductance, gives a limit on the maximum amount of the current that can be transmitted over the line, and by that also the maximum power limit. The capacitive *shunt* component behaves as a reactive power source. The reactive power generated, is proportional to the voltage squared, which implies that the importance of the shunt capacitance increases with the voltage level. For lines having a nominal voltage of 300–500 kV and a length of more than 200 km, these capacitances are of great importance. In high voltage *cables* where the conductors are more close to one another, the capacitance is up to 20–40 times larger than for overhead lines. The reactive power generation can be a problem in cables having a length of only 10 km.

There is a fundamental law about electric fields saying that the electric potential v at a certain point on the distance d from a point charge q, can be calculated as :

$$v = \frac{q}{4\pi\epsilon_0 d} \quad \text{V} \tag{8.75}$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ F/m, permittivity of vacuum. This law gives that there is a direct relationship between the difference in potential and accumulation of charges. If two long, parallel conductors are of interest, and if there is a voltage difference, $v_1 - v_2$, between the lines, an accumulation of charges with different sign, $+Q$ and $-Q$, will take place. The magnitude of the total charge Q depends mainly on the distance between the lines but also on the design of the lines. For cables, the material between the conductors will also have an influence on the charge accumulation. The capacitance between the two conductors is equal to the quotient between the charge Q and the difference in potential :

$$C \equiv \frac{Q}{v_1 - v_2} \tag{8.76}$$

For a three-phase line, the corresponding capacitance is located between all conductors. When having a difference in potential between a conductor and ground, an accumulation of charges will also occur in relation to the magnitude of the capacitance. In Figure 8.9, the different capacitances of a three-phase overhead line are given. A line is normally constructed in a symmetrical way, i.e. the mean distance between the phases are equal. Also, the mean distance between a phase and ground is the same for all phases. In Figure 8.9, this corresponds to the case that $c_{ab} = c_{bc} = c_{ac}$ and $c_{ag} = c_{bg} = c_{cg}$ when the entire line is of interest.

In the same way as given earlier for the series impedances, the positive-, negative- and zero-sequence capacitances can be calculated. Only the results from the calculations will be presented here.

$$C_{-1} = C_{-2} = \frac{2\pi\epsilon_0}{\ln\left[\frac{2Ha}{Ar_{eq}}\right]} \quad \text{F/m} \tag{8.77}$$

$$C_{-0} = \frac{2\pi\epsilon_0}{\ln\left[\frac{2HA^2}{r_{eq}a^2}\right]} \quad \text{F/m} \tag{8.78}$$

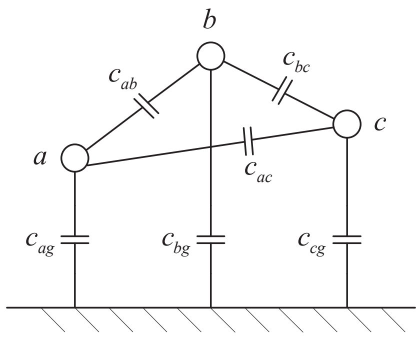


Figure 8.9. Capacitances of a three-phase overhead line without earth wires

where, according to Figure 8.10

$$
\begin{aligned}
C_{-1} &= \text{positive-sequence capacitance} \\
C_{-2} &= \text{negative-sequence capacitance} \\
C_{-0} &= \text{zero-sequence capacitance} \\
H &= \sqrt[3]{H_1 H_2 H_3} \\
A &= \sqrt[3]{A_1 A_2 A_3} \\
a &= \sqrt[3]{a_{12} a_{13} a_{23}} \\
r_{eq} &= \text{the equivalent radius of the line } = e^{-1/4} \times \text{real radius of the line}
\end{aligned}
$$

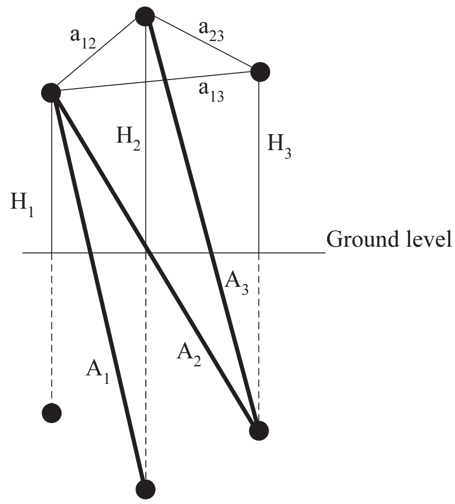


Figure 8.10. Geometrical quantities of a line in the calculation of capacitance

Note that C_1 is equal to C_2, but C_0 has a different value. When having a closer look at the equations for C_1, it can be seen that $2H/A \approx 1$ according to Figure 8.10, which means that the distance to ground has a relatively small influence. If the conductors are located close to one another, then $2H = A$. The line model described in section 6.1.2, uses only the positive-sequence capacitance C_1 for the line. In principle, this can be regarded as a Δ-Y-transformation of the capacitances between the phases since they are the main contributors to the positive- and negative-sequence capacitances. The coupling to ground is of less importance. In cables, the positive- and negative-sequence capacitances are usually higher owing to the short distance between the phases.

For C_{-0}, the coupling to ground is very important. When calculating C_{-0}, all phases have the same potential by the definition of zero-sequence. This implies that the capacitances between the phases c_{ab}, c_{bc}, c_{ac} are not of interest. However, the electric field is changed since all three conductors have the same potential. As given in the equation, the distance to ground is very important (power of three inside the ln-sign) in the calculations of the zero-sequence capacitance.

8.3 Analysis of unbalanced three-phase systems

As discussed in section 8.2, lines and transformers can be represented by their positive-, negative- and zero-sequence impedances. These sequences are decoupled which implies that for instance a certain zero-sequence current will only cause a zero-sequence voltage drop whereas positive- and negative-sequence voltages will be unchanged. Also three-phase generators can in an equivalent way be described by decoupled positive-, negative- and zero-sequence systems.

This property implies that the entire system including generators, lines and transformers can be represented by three decoupled systems.

8.3.1 Connection to a system under unbalanced conditions

In subsection 6.1.3, the connection to a network under symmetrical (or balanced) conditions was discussed. It has been shown that by applying the Thévenin theorem the entire linear balanced system (as seen from a selected point) can be represented by a voltage source behind an impedance. The value of the impedance can be calculated when knowing the three-phase short circuit current at the selected point.

A **balanced** power system as seen from a selected point **p** can be described by three decoupled single-line sequence systems (or networks) termed as positive-, negative and zero-sequence systems. The model of the positive-sequence system is indeed the single-line system of a balanced three-phase system that has been studied in chapters 5-6, i.e. in these chapters we have studied the positive-sequence system of a balanced three-phase system.

Assuming a linear balanced three-phase power system, the sequence systems can be represented by their Thévenin equivalents as shown in Figure 8.11. Note that there are no voltage sources in the network for the negative- and zero-sequence systems. Thus, the negative- and zero-sequence systems only consist of impedances.

From Figure 8.11, the following can be obtained in pu:

$$\begin{aligned}\overline{U}_{p-1} &= \overline{U}_{Thp} - \overline{Z}_{Thp-1}\,\overline{I}_{p-1} \\ \overline{U}_{p-2} &= 0 - \overline{Z}_{Thp-2}\,\overline{I}_{p-2} \\ \overline{U}_{p-0} &= 0 - \overline{Z}_{Thp-0}\,\overline{I}_{p-0}\end{aligned} \tag{8.79}$$

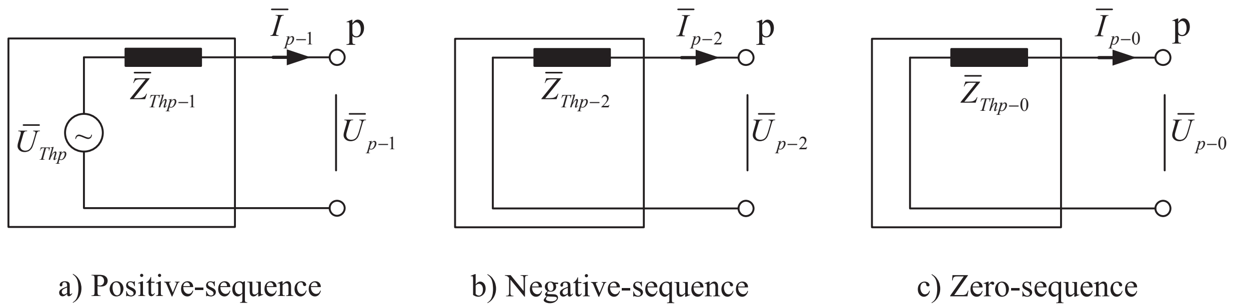


Figure 8.11. Thévenin equivalents as seen from a selected point **p** in the system.

8.3.2 Single line-to-ground fault

Assume that a single line-to-ground fault through an impedance $\overline{Z}_f$ occurs at a point **p** in the system, as shown in Figure 8.12.

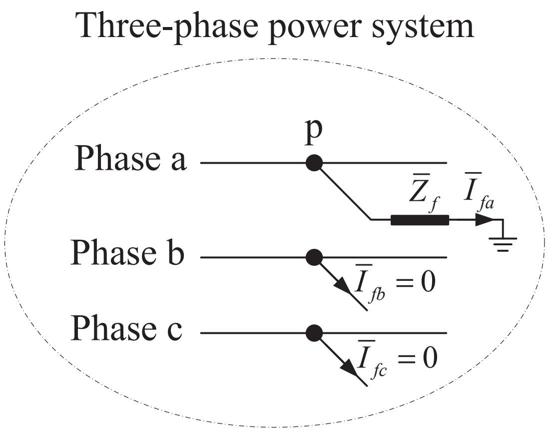


Figure 8.12. Single-phase short circuit in phase a

Based on equation (8.11) the following is obtained:

$$\mathbf{I_s} = \begin{bmatrix} \overline{I}_{p-1} \\ \overline{I}_{p-2} \\ \overline{I}_{p-0} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \overline{I}_{fa} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \overline{I}_{fa} \\ \overline{I}_{fa} \\ \overline{I}_{fa} \end{bmatrix}$$
$$\Rightarrow \quad \overline{I}_{p-1} = \overline{I}_{p-2} = \overline{I}_{p-0} = \frac{1}{3}\overline{I}_{fa} \tag{8.80}$$

From Figure 8.12, we have $\overline{U}_{pa} = \overline{Z}_f \overline{I}_{fa}$.

Using the first row in equation (8.14) and equations (8.79)-(8.80), the following is obtained in pu:

$$\begin{aligned} \overline{U}_{pa} &= \overline{U}_{p-1} + \overline{U}_{p-2} + \overline{U}_{p-0} = \overline{Z}_f \overline{I}_{fa} = 3\overline{Z}_f \overline{I}_{p-1} \quad \Rightarrow \\ \overline{U}_{pa} &= \overline{U}_{Thp} - \overline{Z}_{Thp-1}\overline{I}_{p-1} - \overline{Z}_{Thp-2}\overline{I}_{p-1} - \overline{Z}_{Thp-0}\overline{I}_{p-1} = 3\overline{Z}_f \overline{I}_{p-1} \quad \text{pu} \end{aligned} \tag{8.81}$$

Thus,

$$\overline{I}_{fa} = 3\,\overline{I}_{p-1} = \frac{3\,\overline{U}_{Thp}}{\overline{Z}_{Thp-1} + \overline{Z}_{Thp-2} + \overline{Z}_{Thp-0} + 3\,\overline{Z}_f} \quad \text{pu} \tag{8.82}$$

If the quantities are expressed in their physical units (i.e. kV, kA and Ω), then

$$\begin{aligned}
\overline{U}_{pa} &= \frac{\overline{U}_{Thp}}{\sqrt{3}} - \overline{Z}_{Thp-1}\,\overline{I}_{p-1} - \overline{Z}_{Thp-2}\,\overline{I}_{p-1} - \overline{Z}_{Thp-0}\,\overline{I}_{p-1} = 3\overline{Z}_f\overline{I}_{p-1} \quad \text{kV} \\
\overline{I}_{fa} &= 3\,\overline{I}_{p-1} = \frac{\frac{3\,\overline{U}_{Thp}}{\sqrt{3}}}{\overline{Z}_{Thp-1} + \overline{Z}_{Thp-2} + \overline{Z}_{Thp-0} + 3\,\overline{Z}_f} \quad \text{kA}
\end{aligned} \tag{8.83}$$

If the equivalent diagram of a line shown in Figure 8.7 is used, and if $\overline{U}_a'$ in the figure is connected to $\overline{U}_0'$ through an impedance $\overline{Z}_f$, then by virtue of equation (8.66) the current $\overline{I}_a$ can be obtained as

$$\overline{I}_a = \frac{\overline{U}_a - \overline{U}_0}{\overline{Z}_\alpha + \overline{Z}_\beta + \overline{Z}_f} = \frac{\overline{U}_a - \overline{U}_0}{\overline{Z}_{-1} + \frac{\overline{Z}_{-0} - \overline{Z}_{-1}}{3} + \overline{Z}_f} = \frac{3(\overline{U}_a - \overline{U}_0)}{2\overline{Z}_{-1} + \overline{Z}_{-0} + 3\overline{Z}_f} \tag{8.84}$$

which is similar to equation (8.83). To calculate the current of a single line-to ground fault, the equivalent in Figure 8.7 can be used if $\overline{Z}_{Thp-1} = \overline{Z}_{Thp-2}$.

Example 8.4 *At a 400 kV bus, a solid three-phase short circuit occurs, giving a fault current of 20 kA per phase. If a solid single line-to-ground fault occurs at the same bus, the fault current will be 15 kA in the faulted phase. The Thévenin impedances in the positive- and negative-sequence systems at the bus can be assumed to be purely reactive and equal. (This is normal for high voltage systems since the dominating impedances origin from lines and transformers which have dominating reactive characteristics, equal for positive- and negative-sequences). Also the zero-sequence impedance can be assumed to be purely reactive. Calculate the Thévenin equivalents for the positive-, negative- and zero-sequences at the fault.*

Solution

A solid short circuit means that $\overline{Z}_f = 0$. Since all impedances are purely inductive, the fault currents will also be inductive, i.e.

$$\overline{I}_{sc_{3\Phi}} = -j20 \text{ kA} \qquad \overline{I}_{sc_{1\Phi}} = -j15 \text{ kA} \tag{8.85}$$

Three-phase fault :

Based on equation (6.25),

$$\overline{Z}_{Th-1} = \overline{Z}_{Th-2} = \frac{\overline{U}_{Th}}{\sqrt{3}\,\overline{I}_{sc_{3\Phi}}} = \frac{400}{\sqrt{3}\cdot(-j20)} = j11.55 \quad \Omega \tag{8.86}$$

Single-phase fault :

From equation (8.83),

$$\overline{Z}_{Th-0} = \frac{3\,\overline{U}_{Th}}{\sqrt{3}\,\overline{I}_{sc_{1\Phi}}} - \overline{Z}_{Th-1} - \overline{Z}_{Th-2} - 3\,\overline{Z}_f = \frac{3\cdot 400}{\sqrt{3}\cdot(-j15)} - 2\cdot(j11.55) = j23.09 \quad \Omega \tag{8.87}$$

8.3.3 Analysis of a linear three-phase system with one unbalanced load

As discussed in subsection 8.3.1, the three sequence systems are decoupled when having a balanced system. However, in the case having unsymmetrical loads, these three sequence systems will not be decoupled.

Assume a linear power system with an unbalanced load. The system is composed of a voltage source, lines and transformers. This system can be analyzed as follows:

1. Draw the impedance diagrams of the positive-, negative and zero-sequence systems, for the entire network excluding the unbalanced load.
2. Find the Thévenin equivalents of the positive-, negative- and zero-sequence systems as seen from the point the unsymmetrical load is located.
3. Calculate the positive-, negative- and zero-sequence currents through the unbalanced load.
4. Calculate the positive-, negative- and zero-sequence voltages at the points of interest.
5. Calculate the positive-, negative- and zero-sequence currents through components that are of interest.
6. Transform those symmetrical components to the phase quantities that are asked for.

The abovementioned points can be treated in different ways which will be shown in the following example.

Example 8.5 *Consider again the system described in Example 6.2. The following additional data is also given:*

- *Transformer is Δ-Y0 connected with* $Y0$ *on the 10 kV-side, and* $\overline{Z}_n = 0$.
- *The zero-sequence impedance of the line is 3 times the positive-sequence impedance, i.e.* $\overline{Z}_{21-0} = 3\,\overline{Z}_{21-1}$.
- *The zero-sequence shunt admittance of the line is 0.5 times the positive-sequence shunt admittance, i.e.* $\overline{y}_{sh-21-0} = 0.5\,\overline{y}_{sh-21-1}$.
- *When the transformer is disconnected from bus 3, a solid (i.e.* $\overline{Z}_f = 0$*) single line-to-ground applied to this bus results in a pure inductive fault current of 0.2 kA.*
- *The positive- and negative-sequence Thévenin impedances of the power system are identical, i.e.* $\overline{Z}_{Th-1} = \overline{Z}_{Th-2}$.
- *The load is Y0-connected with* $\overline{Z}_n = 0$*. Furthermore, half of the normal load connected to phase a is disconnected while the other phases are loaded as normal,* **i.e. it is an unsymmetrical load**.

Calculate the voltage at the industry as well as the power fed by the transformer into the line.

Solution

1) Start with the building of the impedance diagram of the positive-, negative- and zero-sequence for the whole system excluding the unsymmetrical load, see Figure 8.13.

Positive- and negative-sequence components in per-unit values (from the solution to Example 6.2):

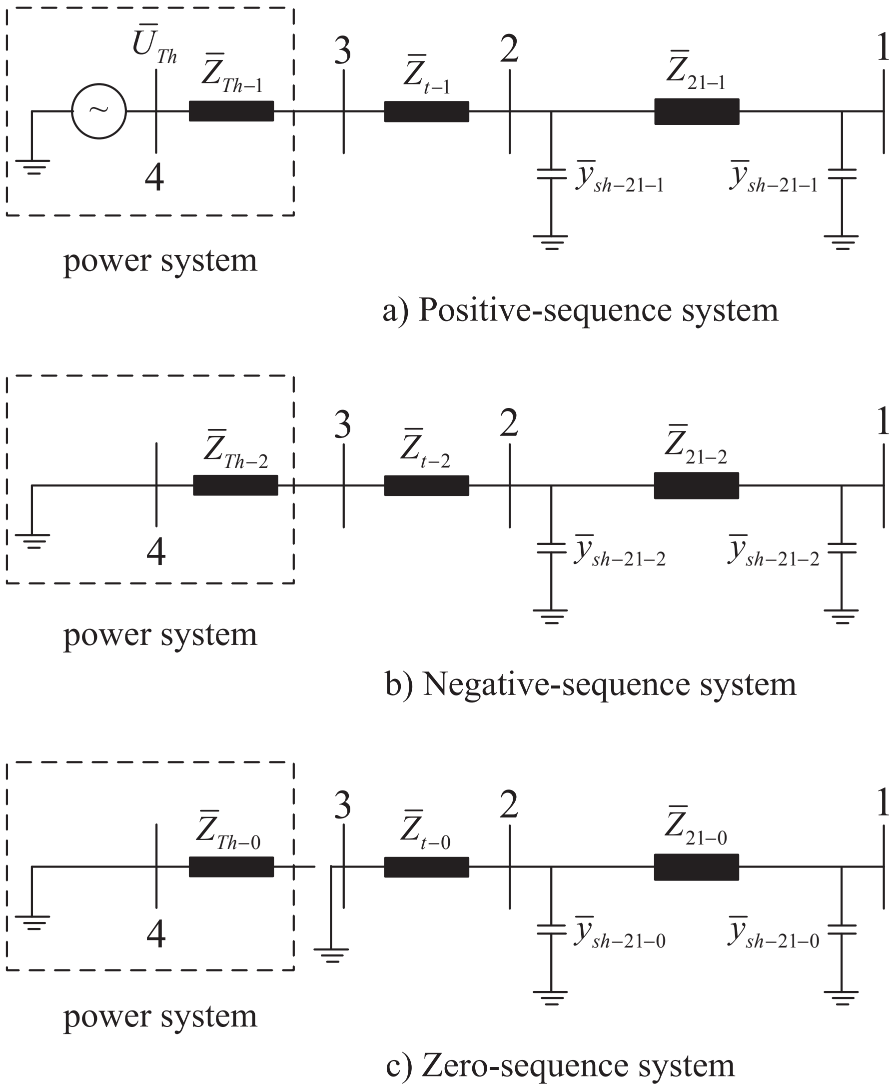


Figure 8.13. Positive-, negative- and zero-sequence systems.

$$\begin{aligned}
\overline{U}_{Th} =& \overline{U}_{Thpu} = 1\angle 0^\circ \\
\overline{Z}_{Th-1} =& \overline{Z}_{Th-2} = \overline{Z}_{Thpu} = j\,0.0137 \\
\overline{Z}_{t-1} =& \overline{Z}_{t-2} = \overline{Z}_{tpu} = j\,0.004 \\
\overline{Z}_{21-1} =& \overline{Z}_{21-2} = \overline{Z}_{21pu} = 0.0225 + j\,0.0075 \\
\overline{y}_{sh-21-1} =& \overline{y}_{sh-21-2} = \overline{y}_{sh-21pu} = \frac{j\,0.003}{2} \\
\overline{A}_{L-1} =& \overline{A}_{L-2} = \overline{A}_L \quad , \quad \overline{B}_{L-1} = \overline{B}_{L-2} = \overline{B}_L \\
\overline{C}_{L-1} =& \overline{C}_{L-2} = \overline{C}_L \quad , \quad \overline{D}_{L-1} = \overline{D}_{L-2} = \overline{D}_L \\
\overline{A}_{-1} =& \overline{A}_{-2} = \overline{A} \quad , \quad \overline{B}_{-1} = \overline{B}_{-2} = \overline{B} \\
\overline{C}_{-1} =& \overline{C}_{-2} = \overline{C} \quad , \quad \overline{D}_{-1} = \overline{D}_{-2} = \overline{D}
\end{aligned} \tag{8.88}$$

Zero-sequence components in per-unit values:

$$\begin{aligned}
\overline{I}_{sc_{1\Phi}} =& \frac{0.2\angle -90^\circ}{I_{b70}} = \frac{0.2\angle -90^\circ}{0.00412} = 48.5437\angle -90^\circ \\
\overline{Z}_{Th-0} =& \frac{3\,\overline{U}_{Th}}{\overline{I}_{sc_{1\Phi}}} - 2\,\overline{Z}_{Th-1} - 0 = j\,0.0344 \\
\overline{Z}_{t-0} =& \overline{Z}_{t-1} = j\,0.004 \quad , \quad \text{since } \overline{Z}_n = 0 \\
\overline{Z}_{21-0} =& 3\,\overline{Z}_{21-1} = 0.0675 + j\,0.0225 \\
\overline{y}_{sh-21-0} =& 0.5\,\overline{y}_{sh-21-1} = \frac{j\,0.003}{4} \\
\overline{A}_{L-0} =& 1 + \overline{y}_{sh-21-0} \cdot \overline{Z}_{21-0} = 1.0000 + j\,0.0001 \\
\overline{B}_{L-0} =& \overline{Z}_{21-0} = 0.0675 + j\,0.0225 \\
\overline{C}_{L-0} =& \overline{y}_{sh-21-0}(2 + \overline{y}_{sh-21-0} \cdot \overline{Z}_{21-0}) = 0.0000 + j\,0.0015 \\
\overline{D}_{L-0} =& \overline{A}_{L-0} = 1.0000 + j\,0.0001
\end{aligned} \tag{8.89}$$

2) Next step is to replace the networks with Thévenin equivalents as seen from the industry connection point (bus 1), i.e. $\overline{U}_{Thbus1}, \overline{Z}_{Thbus1-1}, \overline{Z}_{Thbus1-2}$ and $\overline{Z}_{Thbus1-0}$, see Figure 8.11.

The twoport of the entire positive-sequence network between bus 4 (which represents the voltage source) and bus 1 (the industry connection point) is given by (see also Example 6.2):

$$\begin{aligned}
\begin{bmatrix} \overline{U}_{Th} \\ \overline{I}_{bus4-1} \end{bmatrix} &= \begin{bmatrix} \overline{A}_{-1} & \overline{B}_{-1} \\ \overline{C}_{-1} & \overline{D}_{-1} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-1} \\ \overline{I}_{bus1-1} \end{bmatrix} = \\
&= \begin{bmatrix} 0.9999 + j0.0000 & 0.0225 + j0.0252 \\ 0.0000 + j0.0030 & 1.0000 + j0.0000 \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-1} \\ \overline{I}_{bus1-1} \end{bmatrix}
\end{aligned} \tag{8.90}$$

Based on Figure 8.11 a), $\overline{U}_{Thbus1-1}$ is obtained by setting $\overline{I}_{bus1-1} = 0$ as follows:

$$\begin{aligned}
\overline{U}_{Th} =& \overline{A}_{-1}\,\overline{U}_{Thbus1-1} + \overline{B}_{-1} \cdot 0 \quad \Rightarrow \\
\overline{U}_{Thbus1-1} =& \frac{\overline{U}_{Th}}{\overline{A}_{-1}} = 1.0001\angle -0.0019^\circ
\end{aligned} \tag{8.91}$$

The Thévenin impedance $\overline{Z}_{Thbus1-1}$ is obtained by setting $\overline{U}_{Th} = 0$ as follows:

$$\begin{aligned} 0 &= \overline{A}_{-1}\,\overline{U}_{bus1-1} + \overline{B}_{-1}\,\overline{I}_{bus1-1} \quad \Rightarrow \\ \overline{Z}_{Thbus1-1} &= -\frac{\overline{U}_{bus1-1}}{\overline{I}_{bus1-1}} = \frac{\overline{B}_{-1}}{\overline{A}_{-1}} = 0.0225 + j0.0252 \end{aligned} \tag{8.92}$$

To set $\overline{U}_{Th} = 0$, it means that bus 4 is short circuited. Therefore in this case, the positive-sequence system will have a configuration similar to the negative-sequence system as shown in Figure 8.11 b). Furthermore, since $\overline{Z}_{t-1} = \overline{Z}_{t-2}$, $\overline{Z}_{12-1} = \overline{Z}_{12-2}$ and $\overline{y}_{sh-21-1} = \overline{y}_{sh-21-2}$, they imply that $\overline{A}_{-1} = \overline{A}_{-2}$ and $\overline{B}_{-1} = \overline{B}_{-2}$. Thus, in a similar way as shown in equation (8.92), the Thévenin impedance $\overline{Z}_{Thbus1-2}$ is obtained as follows:

$$\begin{aligned} 0 &= \overline{A}_{-2}\,\overline{U}_{bus1-2} + \overline{B}_{-2}\,\overline{I}_{bus1-2} \quad \Rightarrow \\ \overline{Z}_{Thbus1-2} &= -\frac{\overline{U}_{bus1-2}}{\overline{I}_{bus1-2}} = \frac{\overline{B}_{-2}}{\overline{A}_{-2}} = \frac{\overline{B}_{-1}}{\overline{A}_{-1}} = \overline{Z}_{Thbus1-1} = 0.0225 + j0.0252 \end{aligned} \tag{8.93}$$

Based on Figure 8.3 d), the zero-sequence of a Δ-Y0-transformer should be modeled as an impedance to ground on the $Y0$-side, as shown in Figure 8.13 c). As seen in the figure, the feeding network (i.e. power system) is not connected to the industry load from a zero-sequence point of view. The twoport of the network from the transformer (bus 3) to the connection point of the industry (bus 1) is given by

$$\begin{aligned} \begin{bmatrix} \overline{U}_{bus3-0} \\ \overline{I}_{bus3-0} \end{bmatrix} &= \begin{bmatrix} 0 \\ \overline{I}_{bus3-0} \end{bmatrix} = \begin{bmatrix} 1 & \overline{Z}_{t-0} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{A}_{L-0} & \overline{B}_{L-0} \\ \overline{C}_{L-0} & \overline{D}_{L-0} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-0} \\ \overline{I}_{bus1-0} \end{bmatrix} = \\ &= \begin{bmatrix} \overline{A}_{-0} & \overline{B}_{-0} \\ \overline{C}_{-0} & \overline{D}_{-0} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-0} \\ \overline{I}_{bus1-0} \end{bmatrix} = \\ &= \begin{bmatrix} 1.0000 + j0.0001 & 0.0675 + j0.0265 \\ 0.0000 + j0.0015 & 1.0000 + j0.0001 \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-0} \\ \overline{I}_{bus1-0} \end{bmatrix} \end{aligned} \tag{8.94}$$

The Thévenin impedance $\overline{Z}_{Thbus1-0}$ is obtained as follows:

$$\begin{aligned} 0 &= \overline{A}_{-0}\,\overline{U}_{bus1-0} + \overline{B}_{-0}\,\overline{I}_{bus1-0} \quad \Rightarrow \\ \overline{Z}_{Thbus1-0} &= -\frac{\overline{U}_{bus1-0}}{\overline{I}_{bus1-0}} = \frac{\overline{B}_{-0}}{\overline{A}_{-0}} = 0.0675 + j0.0265 \end{aligned} \tag{8.95}$$

3) From Example 6.2 the per-unit value of the load (i.e. $\overline{Z}_{LD}$) is known. At the half load in phase a (i.e. $\overline{Z}_{LDa} = 2\overline{Z}_{LD}$), the impedance matrix of the symmetrical components is calculated based on equation (8.34) :

$$\begin{aligned} \mathbf{Z_{LDs}} &= \mathbf{T^{-1}}\,\mathbf{Z_{LDPh}}\,\mathbf{T} = \mathbf{T^{-1}} \begin{bmatrix} 2\overline{Z}_{LD} & 0 & 0 \\ 0 & \overline{Z}_{LD} & 0 \\ 0 & 0 & \overline{Z}_{LD} \end{bmatrix} \mathbf{T} \\ &= \begin{bmatrix} 1.0667 + j0.8000 & 0.2667 + j0.2000 & 0.2667 + j0.2000 \\ 0.2667 + j0.2000 & 1.0667 + j0.8000 & 0.2667 + j0.2000 \\ 0.2667 + j0.2000 & 0.2667 + j0.2000 & 1.0667 + j0.8000 \end{bmatrix} \end{aligned} \tag{8.96}$$

The equation of the entire system is now given by

$$\mathbf{U_{Th}} = \begin{bmatrix} \overline{U}_{Thbus1} \\ 0 \\ 0 \end{bmatrix} = (\mathbf{Z_s} + \mathbf{Z_{LDs}})\,\mathbf{I_s} \tag{8.97}$$

where

$$\mathbf{Z_s} = \begin{bmatrix} \overline{Z}_{Thbus1-1} & 0 & 0 \\ 0 & \overline{Z}_{Thbus1-2} & 0 \\ 0 & 0 & \overline{Z}_{Thbus1-0} \end{bmatrix} \quad \text{and} \quad \mathbf{I_s} = \begin{bmatrix} \overline{I}_{bus1-1} \\ \overline{I}_{bus1-2} \\ \overline{I}_{bus1-0} \end{bmatrix}$$

The symmetrical components of the currents through the load can be calculated as:

$$\mathbf{I_s} = (\mathbf{Z_s} + \mathbf{Z_{LDs}})^{-1}\,\mathbf{U_{Th}} = \begin{bmatrix} 0.8084\angle-37.1616^\circ \\ 0.1596\angle142.3433^\circ \\ 0.1541\angle143.7484^\circ \end{bmatrix} \tag{8.98}$$

4-5) The symmetrical components of the voltage at the industry (bus 1) are given by

$$\mathbf{U_{bus1s}} = \begin{bmatrix} \overline{U}_{bus1-1} \\ \overline{U}_{bus1-2} \\ \overline{U}_{bus1-0} \end{bmatrix} = \mathbf{Z_{LDs}}\,\mathbf{I_s} = \begin{bmatrix} 0.9733\angle-0.3126^\circ \\ 0.0054\angle10.6340^\circ \\ 0.0112\angle-14.8199^\circ \end{bmatrix} \tag{8.99}$$

The positive-, negative- and zero-sequence voltages and currents (in per-unit values) at the transformer connection to the line (bus 2) are given by

$$\begin{aligned} \begin{bmatrix} \overline{U}_{bus2-1} \\ \overline{I}_{bus2-1} \end{bmatrix} &= \begin{bmatrix} \overline{A}_{L-1} & \overline{B}_{L-1} \\ \overline{C}_{L-1} & \overline{D}_{L-1} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-1} \\ \overline{I}_{bus1-1} \end{bmatrix} = \begin{bmatrix} 0.9915\angle-0.6607^\circ \\ 0.8066\angle-36.9937^\circ \end{bmatrix} \\ \begin{bmatrix} \overline{U}_{bus2-2} \\ \overline{I}_{bus2-2} \end{bmatrix} &= \begin{bmatrix} \overline{A}_{L-2} & \overline{B}_{L-2} \\ \overline{C}_{L-2} & \overline{D}_{L-2} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-2} \\ \overline{I}_{bus1-2} \end{bmatrix} = \begin{bmatrix} 0.0028\angle52.3414^\circ \\ 0.1596\angle142.3414^\circ \end{bmatrix} \\ \begin{bmatrix} \overline{U}_{bus2-0} \\ \overline{I}_{bus2-0} \end{bmatrix} &= \begin{bmatrix} \overline{A}_{L-0} & \overline{B}_{L-0} \\ \overline{C}_{L-0} & \overline{D}_{L-0} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-0} \\ \overline{I}_{bus1-0} \end{bmatrix} = \begin{bmatrix} 0.0006\angle53.7455^\circ \\ 0.1541\angle143.7455^\circ \end{bmatrix} \end{aligned} \tag{8.100}$$

The symmetrical components of power (in physical units) fed by the transformer into the line can be expressed by

$$\begin{aligned} \overline{S}_{bus2-1} &= \overline{U}_{bus2-1}\,\overline{I}^{*}_{bus2-1}\,S_{base} = 0.3221 + j\,0.2369 \text{ MVA} \\ \overline{S}_{bus2-2} &= \overline{U}_{bus2-2}\,\overline{I}^{*}_{bus2-2}\,S_{base} = 0 - j\,0.0002 \text{ MVA} \\ \overline{S}_{bus2-0} &= \overline{U}_{bus2-0}\,\overline{I}^{*}_{bus2-0}\,S_{base} = 0 - j\,0.00005 \text{ MVA} \end{aligned} \tag{8.101}$$

6) Based on equation (8.14), the line-to-neutral voltages can be obtained. To express these quantities in physical units, they must be multiplied with the corresponding base voltage, then divided by $\sqrt{3}$, since the base voltage is based on line-to-line voltage. Thus,

$$\begin{bmatrix} \overline{U}_{bus1a} \\ \overline{U}_{bus1b} \\ \overline{U}_{bus1c} \end{bmatrix} = \mathbf{T}\,\mathbf{U_{bus1s}} \cdot \frac{U_{base10}}{\sqrt{3}} = \begin{bmatrix} 5.7122\angle-0.4154^\circ \\ 5.5918\angle-119.9774^\circ \\ 5.5535\angle119.4555^\circ \end{bmatrix} \tag{8.102}$$

Based on equation (8.26), the total power fed by the transformer into the line is given by

$$\overline{S} = \overline{S}_{bus2-1} + \overline{S}_{bus2-2} + \overline{S}_{bus2-0} = 0.3221 + j0.2366 \text{ MVA} \tag{8.103}$$

8.4 A general method for analysis of linear three-phase systems with one unbalanced load

In larger unsymmetrical systems, it is necessary to use a systematic approach to analyze system voltages and currents. In this section, all system components except one load, are symmetrical. In the demonstration below, a small system is analyzed in the same way as can be performed for a large system. The example given below is identical to the one in section 6.2 but with the difference that the load which will be connected to bus 2 is assumed unbalanced. The voltage source is represented by bus 3. All quantities are expressed in per-unit values.

Consider the simple balanced system shown in Figure 8.14. For a balanced system, all system quantities and components can be represented only by their positive-sequence components. The only difference between this system and the system studied in section 6.2 is that here we use index -1 which has been omitted in section 6.2.

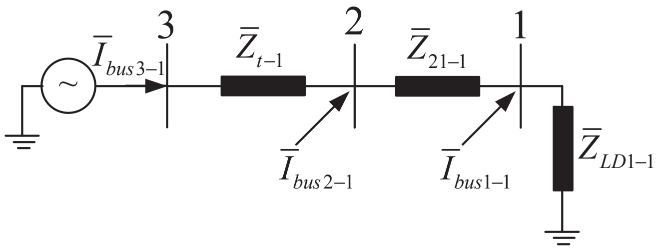


Figure 8.14. Single-phase impedance diagram of a symmetrical system.

The Y-bus matrix of the positive-sequence system is identical with Y in section 6.2, i.e.

$$\begin{bmatrix} \overline{I}_{bus1-1} \\ \overline{I}_{bus2-1} \\ \overline{I}_{bus3-1} \end{bmatrix} = \mathbf{I_1} = \mathbf{Y_1}\,\mathbf{U_1} = \\ = \begin{bmatrix} \frac{1}{\overline{Z}_{LD1-1}} + \frac{1}{\overline{Z}_{12-1}} & -\frac{1}{\overline{Z}_{12-1}} & 0 \\ -\frac{1}{\overline{Z}_{21-1}} & \frac{1}{\overline{Z}_{21-1}} + \frac{1}{\overline{Z}_{t-1}} & -\frac{1}{\overline{Z}_{t-1}} \\ 0 & -\frac{1}{\overline{Z}_{t-1}} & \frac{1}{\overline{Z}_{t-1}} \end{bmatrix} \begin{bmatrix} \overline{U}_{bus1-1} \\ \overline{U}_{bus2-1} \\ \overline{U}_{bus3-1} \end{bmatrix} \tag{8.104}$$

This Y-bus matrix can be inverted which results in the corresponding Z-bus matrix :

$$\mathbf{U_1} = \mathbf{Y_1^{-1}}\,\mathbf{I_1} = \mathbf{Z_1}\,\mathbf{I_1} \tag{8.105}$$

Since $\overline{I}_{bus1-1} = \overline{I}_{bus2-1} = 0$, the third row in equation (8.105) can be written as

$$\begin{aligned} \overline{U}_{bus3-1} &= \mathbf{Z_1}(3,3) \cdot \overline{I}_{bus3-1} \quad \Rightarrow \\ \overline{I}_{bus3-1} &= \frac{\overline{U}_{bus3-1}}{\mathbf{Z_1}(3,3)} \end{aligned} \tag{8.106}$$

where $\mathbf{Z_1}(3,3)$ is an element in the Z-bus matrix. With this value of the current inserted into equation (8.105) all voltages are obtained.

$$\overline{U}_{bus1-1} = \mathbf{Z_1}(1,3) \cdot \overline{I}_{bus3-1} \tag{8.107}$$

$$\overline{U}_{bus2-1} = \mathbf{Z_1}(2,3) \cdot \overline{I}_{bus3-1} \tag{8.108}$$

So far, all calculations are identical to those in section 6.2. Corresponding calculations can be performed for an arbitrary large system having **impedance loads and one voltage source**. Assume a system with a voltage source at bus i. The current at bus i and the voltage at another bus r can then be calculated as:

$$\overline{I}_{busi-1} = \frac{\overline{U}_{busi-1}}{\mathbf{Z_1}(i,i)} \tag{8.109}$$

$$\overline{U}_{busr-1} = \overline{U}_{Thbusr} = \mathbf{Z_1}(r,i) \cdot \overline{I}_{busi-1} \tag{8.110}$$

Now assume that an unsymmetrical impedance load is connected to bus 2 which apparently leads to the changes of the system voltages and currents. The actual voltages can be obtained by using the theorem of superposition, i.e. as the sum of the voltages before the connection of the load and with the voltage change obtained by the load connection. This can be expressed by using symmetrical components as:

$$\begin{aligned} \mathbf{U'_1} &= \mathbf{U_{pre1}} + \mathbf{U_{\Delta 1}} \\ \mathbf{U'_2} &= \mathbf{U_{pre2}} + \mathbf{U_{\Delta 2}} \\ \mathbf{U'_0} &= \mathbf{U_{pre0}} + \mathbf{U_{\Delta 0}} \end{aligned} \tag{8.111}$$

where, (**below with all buses it means all buses in the system excluding the bus connected to the voltage source**)

$\mathbf{U'_1}$ is a vector containing the positive-sequence voltages at all buses (i.e. bus 1 and bus 2 in this example) due to the connection of the unsymmetrical load.

$\mathbf{U'_2}$ is a vector containing the negative-sequence voltages at all buses due to the connection of the unsymmetrical load.

$\mathbf{U'_0}$ is a vector containing the zero-sequence voltages at all buses due to the connection of the unsymmetrical load.

$\mathbf{U_{pre1}}$ is a vector containing the positive-sequence voltages at all buses prior to the connection of the unsymmetrical load.

$\mathbf{U_{pre2}}$ is a vector containing the negative-sequence voltages at all buses prior to the connection of the unsymmetrical load. All elements of this vector are zero, since the system is under balanced conditions prior to the connection of the unsymmetrical load.

$\mathbf{U_{pre0}}$ is a vector containing the zero-sequence voltages at all buses prior to the connection of the unsymmetrical load. All elements of this vector are zero, since the system is under balanced conditions prior to the connection of the unsymmetrical load.

$\mathbf{U_{\Delta 1}}$ is a vector containing the changes in the positive-sequence voltages at all buses due to the connection of the unsymmetrical load.

$\mathbf{U_{\Delta 2}}$ is a vector containing the changes in the negative-sequence voltages at all buses due to the connection of the unsymmetrical load.

$\mathbf{U_{\Delta 0}}$ is a vector containing the changes in the zero-sequence voltages at all buses due to the connection of the unsymmetrical load.

Equation (8.111) can be rewritten by expressing the voltage changes by a Z-bus matrix multiplied with the current changes injected into the buses as follows:

$$\begin{aligned} \mathbf{U_1'} &= \mathbf{U_{pre1}} + \mathbf{Z_{\Delta 1}}\,\mathbf{I_{\Delta 1}} \\ \mathbf{U_2'} &= \quad \mathbf{0} \quad + \mathbf{Z_{\Delta 2}}\,\mathbf{I_{\Delta 2}} \\ \mathbf{U_0'} &= \quad \mathbf{0} \quad + \mathbf{Z_{\Delta 0}}\,\mathbf{I_{\Delta 0}} \end{aligned} \tag{8.112}$$

where

$\mathbf{Z_{\Delta 1}}$ is the Z-bus matrix of the positive-sequence system with the shortened voltage source.

$\mathbf{Z_{\Delta 2}}$ is the Z-bus matrix of the negative-sequence system.

$\mathbf{Z_{\Delta 0}}$ is the Z-bus matrix of the zero-sequence system.

$\mathbf{I_{\Delta 1}}$ is a vector containing the injected positive-sequence current changes into the buses. In this example only $\mathbf{I_{\Delta 1}}(2) \neq 0$, since the load is connected to bus 2.

$\mathbf{I_{\Delta 2}}$ is a vector containing the injected negative-sequence current changes into the buses. In this example only $\mathbf{I_{\Delta 2}}(2) \neq 0$.

$\mathbf{I_{\Delta 0}}$ is a vector containing the injected zero-sequence current changes into the buses. In this example only $\mathbf{I_{\Delta 0}}(2) \neq 0$.

Figure 8.15 shows the positive-, negative- and zero-sequence systems which will be used to calculate the voltage changes due to connection of the unsymmetrical load at bus 2. The difference between the positive-sequence system in Figure 8.15 and the Δ-system used in section 6.2 is that the load is now represented by the currents injected into the buses. The infinite bus (bus 1) is assumed to be directly connected to ground and the transformer is Y0-Y0 connected. The admittance matrices of the sequence networks shown in Figure 8.15 can be formed as

$$\begin{aligned} \mathbf{Y_{\Delta 1}} &= \begin{bmatrix} \frac{1}{\overline{Z}_{LD1-1}} + \frac{1}{\overline{Z}_{21-1}} & -\frac{1}{\overline{Z}_{21-1}} \\ -\frac{1}{\overline{Z}_{21-1}} & \frac{1}{\overline{Z}_{21-1}} + \frac{1}{\overline{Z}_{t-1}} \end{bmatrix} \\ \mathbf{Y_{\Delta 2}} &= \begin{bmatrix} \frac{1}{\overline{Z}_{LD1-2}} + \frac{1}{\overline{Z}_{21-2}} & -\frac{1}{\overline{Z}_{21-2}} \\ -\frac{1}{\overline{Z}_{21-2}} & \frac{1}{\overline{Z}_{21-2}} + \frac{1}{\overline{Z}_{t-2}} \end{bmatrix} \\ \mathbf{Y_{\Delta 0}} &= \begin{bmatrix} \frac{1}{\overline{Z}_{LD1-0}} + \frac{1}{\overline{Z}_{21-0}} & -\frac{1}{\overline{Z}_{21-0}} \\ -\frac{1}{\overline{Z}_{21-0}} & \frac{1}{\overline{Z}_{21-0}} + \frac{1}{\overline{Z}_{t-0}} \end{bmatrix} \end{aligned} \tag{8.113}$$

Note that $\mathbf{Y_{\Delta 1}} = \mathbf{Y_1}(1:2, 1:2)$, i.e. the row and column corresponding to the bus connected to the voltage source (bus 1 in this example) are removed, see also $\mathbf{Y_\Delta}$ in section

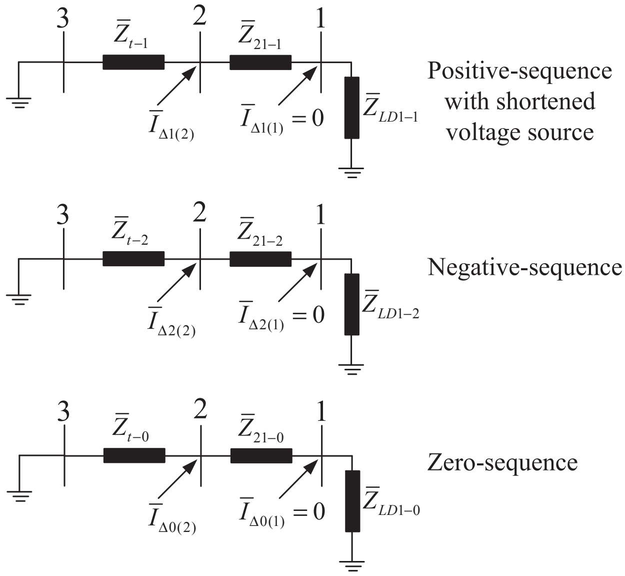


Figure 8.15. Positive-, negative- and zero-sequence diagrams for calculations of the voltage changes.

6.2. Furthermore, $\mathbf{Y_{\Delta 2}} = \mathbf{Y_{\Delta 1}}$ for a system that is only composed of lines, transformers and symmetrical impedance loads, since their positive- and negative-sequence components are identical.

From the above Y-bus matrices the corresponding Z-bus matrices can be calculated as

$$\begin{aligned} \mathbf{Z_{\Delta 1}} &= \mathbf{Y_{\Delta 1}^{-1}} \\ \mathbf{Z_{\Delta 2}} &= \mathbf{Y_{\Delta 2}^{-1}} \\ \mathbf{Z_{\Delta 0}} &= \mathbf{Y_{\Delta 0}^{-1}} \end{aligned} \tag{8.114}$$

Since only the sequence components of the injected currents (i.e. $\mathbf{I_\Delta}$) into the bus to which the unsymmetrical load is connected (bus 2 in this example) are nonzero, these currents are of interest and will be calculated as follows. Based on equation (8.112), we have

$$\begin{aligned} \overline{U}'_{bus2-1} &= \mathbf{U'_1}(2) = \underbrace{\overline{U}_{bus2-1}}_{\text{from eq. (8.110)}} + \mathbf{Z_{\Delta 1}}(2,2)\,\mathbf{I_{\Delta 1}}(2) \\ \overline{U}'_{bus2-2} &= \mathbf{U'_2}(2) = 0 + \mathbf{Z_{\Delta 2}}(2,2)\,\mathbf{I_{\Delta 2}}(2) \\ \overline{U}'_{bus2-0} &= \mathbf{U'_0}(2) = 0 + \mathbf{Z_{\Delta 0}}(2,2)\,\mathbf{I_{\Delta 0}}(2) \end{aligned} \tag{8.115}$$

Assuming that the unsymmetrical load is connected to bus r ($r = 2$ in this example), the equations can be summarized as

$$\begin{aligned} \mathbf{U'_s}(r) &= \begin{bmatrix} \overline{U}'_{busr-1} \\ \overline{U}'_{busr-2} \\ \overline{U}'_{busr-0} \end{bmatrix} = \mathbf{U_{pre}}(r) + \mathbf{Z_\Delta}(r,r)\,\mathbf{I_\Delta}(r) = \\ &\equiv \begin{bmatrix} \overline{U}_{Thbusr} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{Z_{\Delta 1}}(r,r) & 0 & 0 \\ 0 & \mathbf{Z_{\Delta 2}}(r,r) & 0 \\ 0 & 0 & \mathbf{Z_{\Delta 0}}(r,r) \end{bmatrix} \begin{bmatrix} \overline{I}_{\Delta 1}(r) \\ \overline{I}_{\Delta 2}(r) \\ \overline{I}_{\Delta 0}(r) \end{bmatrix} \end{aligned} \tag{8.116}$$

It should be pointed out that equation (8.116) indeed describes the Thévenin equivalents as seen from bus r, where the voltage behind the positive-sequence impedance is $\overline{U}_{Thbusr} = \overline{U}_{bus2-1}$ and the three Thévenin impedances are $\mathbf{Z_{\Delta 1}}(r,r)$, $\mathbf{Z_{\Delta 2}}(r,r)$ and $\mathbf{Z_{\Delta 0}}(r,r)$.

Assume that the unsymmetrical load is Y0-connected with $\overline{Z}_{LDbusra}$ in phase a, $\overline{Z}_{LDbusrb}$ in phase b and $\overline{Z}_{LDbusrc}$ in phase c. The voltage drop over the load is

$$\begin{aligned}
\mathbf{U_{LDbusrp_h}} &= \begin{bmatrix} \overline{U}_{LDbusra} \\ \overline{U}_{LDbusrb} \\ \overline{U}_{LDbusrc} \end{bmatrix} = \begin{bmatrix} \overline{Z}_{LDbusra} & 0 & 0 \\ 0 & \overline{Z}_{LDbusrb} & 0 \\ 0 & 0 & \overline{Z}_{LDbusrc} \end{bmatrix} \begin{bmatrix} \overline{I}_{LDbusra} \\ \overline{I}_{LDbusrb} \\ \overline{I}_{LDbusrc} \end{bmatrix} = \\
&= \mathbf{Z_{LDbusrp_h}} \, \mathbf{I_{LDbusrp_h}}
\end{aligned} \tag{8.117}$$

By introducing symmetrical components, this can be converted to

$$\begin{aligned}
\mathbf{U'_s}(r) &= \begin{bmatrix} \overline{U}'_{busr-1} \\ \overline{U}'_{busr-2} \\ \overline{U}'_{busr-0} \end{bmatrix} = \mathbf{T^{-1}} \, \mathbf{U_{LDbusrp_h}} = \mathbf{T^{-1}} \, \mathbf{Z_{LDbusrp_h}} \, \mathbf{I_{LDbusrp_h}} = \\
&= \underbrace{\mathbf{T^{-1}} \, \mathbf{Z_{LDbusrp_h}} \, \mathbf{T}}_{=\mathbf{Z_{LDbusrs}}} \mathbf{I_{LDbusrs}} = -\mathbf{Z_{LDbusrs}} \, \mathbf{I_{\Delta}}(r)
\end{aligned} \tag{8.118}$$

Note that $\mathbf{I_{LDbusrs}}$ is injected into the load, however $\mathbf{I_{\Delta}}(r)$ is injected into the bus. Therefore, $\mathbf{I_{LDbusrs}} = -\mathbf{I_{\Delta}}(r)$.

Next based on equations (8.116) and (8.118), the current $\mathbf{I_{\Delta}}(r)$ can be expressed by

$$\mathbf{I_{\Delta}}(r) = -\left[\mathbf{Z_{\Delta}}(r,r) + \mathbf{Z_{LDbusrs}}\right]^{-1} \mathbf{U_{pre}}(r) \tag{8.119}$$

These values of the symmetrical components of the current at bus r can then be inserted into equation (8.112) where the symmetrical components of all voltages can be calculated. The voltage at bus k can then be calculated as :

$$\begin{aligned}
\mathbf{U'_1}(k) &= \mathbf{U_{pre1}}(k) + \mathbf{Z_{\Delta 1}}(k,r) \, \mathbf{I_{\Delta 1}}(r) \\
\mathbf{U'_2}(k) &= \mathbf{Z_{\Delta 2}}(k,r) \, \mathbf{I_{\Delta 2}}(r) \\
\mathbf{U'_0}(k) &= \mathbf{Z_{\Delta 0}}(k,r) \, \mathbf{I_{\Delta 0}}(r)
\end{aligned} \tag{8.120}$$

Example 8.6 *Consider again the system described in Example 6.3. The following additional data is also given:*

- *The infinite bus (i.e. bus 1) has a grounded zero connection point.*
- *Transformer T1 is Δ-Y0 connected with* $Y0$ *on the 10 kV-side, and* $\overline{Z}_n = 0$.
- *Transformer T2 is Y0-Y0 connected with* $\overline{Z}_n = 0$.
- *The zero-sequence impedances of the lines are 3 times the positive-sequence impedances, and the zero-sequence shunt admittances of the lines are 0.5 times the positive-sequence shunt admittances.*
- *The load LD1 is Δ-connected.*

- *The load LD2 is Y0-connected with $\overline{Z}_n = 0$. Furthermore, half of the normal load connected to phase a is disconnected while the other phases are loaded as normal,* **i.e. LD2 is an unsymmetrical load**.

Calculate the efficiency of the internal network operating in this unbalanced condition.
Solution

1) Start with the building of the impedance diagram of the positive-, negative- and zero-sequence networks for the entire system excluding the unsymmetrical load, see Figure 8.16.

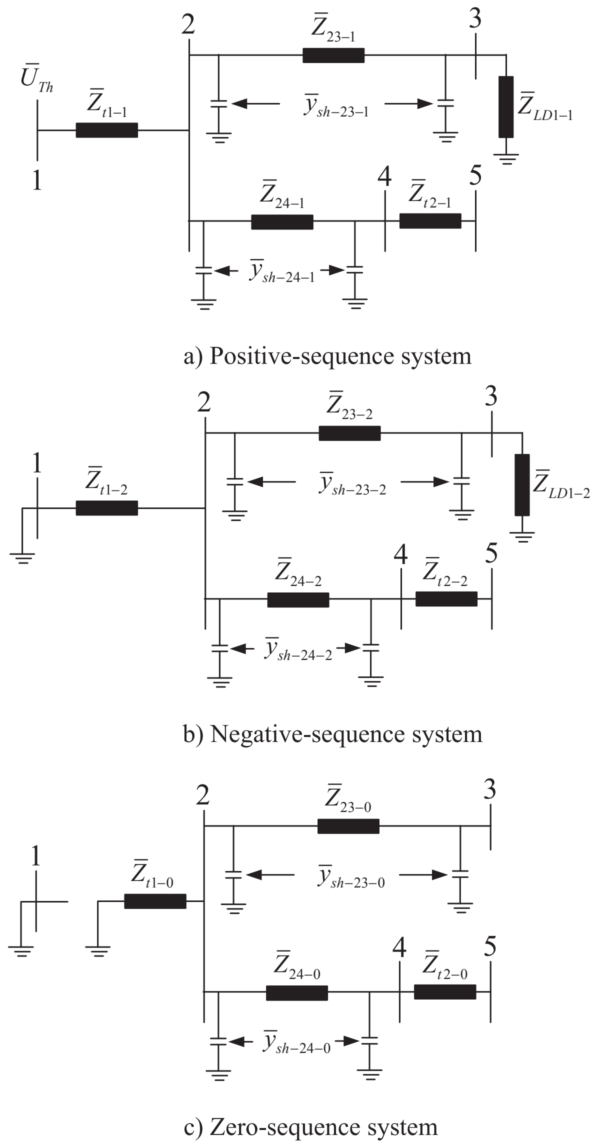


Figure 8.16. The sequence networks of the system in Example 8.6.

Positive- and negative-sequence components in per-unit values (from the solution to Example 6.3):

$$\begin{aligned}
\overline{U}_{Th} &= \overline{U}_1 = 1\angle 0^\circ \\
\overline{Z}_{t1-1} &= \overline{Z}_{t1-2} = \overline{Z}_{t1pu} = j\,0.0438 \\
\overline{Z}_{t2-1} &= \overline{Z}_{t2-2} = \overline{Z}_{t2pu} = j\,0.1333 \\
\overline{Z}_{23-1} &= \overline{Z}_{23-2} = \overline{Z}_{23pu} = 0.0017 + j\,0.003 \\
\overline{y}_{sh-23-1} &= \overline{y}_{sh-23-2} = \overline{y}_{sh-23pu} = \frac{j\,0.0013}{2} \\
\overline{Z}_{24-1} &= \overline{Z}_{24-2} = \overline{Z}_{24pu} = 0.0009 + j\,0.0015 \\
\overline{y}_{sh-24-1} &= \overline{y}_{sh-24-2} = \overline{y}_{sh-24pu} = \frac{j\,0.00064}{2} \\
\overline{Z}_{LD1-1} &= \overline{Z}_{LD1-2} = \overline{Z}_{LD1pu} = 0.64 + j\,0.48
\end{aligned} \tag{8.121}$$

Zero-sequence components in per-unit values:

$$\begin{aligned}
\overline{Z}_{t1-0} &= \overline{Z}_{t1-1} = j\,0.0438 \quad , \quad \text{since } \overline{Z}_n = 0 \\
\overline{Z}_{t2-0} &= \overline{Z}_{t2-1} = j\,0.1333 \quad , \quad \text{since } \overline{Z}_n = 0 \\
\overline{Z}_{23-0} &= 3\,\overline{Z}_{23-1} = 0.0051 + j\,0.009 \\
\overline{y}_{sh-23-0} &= 0.5\,\overline{y}_{sh-23-1} = \frac{j\,0.0013}{4} \\
\overline{Z}_{24-0} &= 3\,\overline{Z}_{24-1} = 0.0026 + j\,0.0045 \\
\overline{y}_{sh-24-0} &= 0.5\,\overline{y}_{sh-24-1} = \frac{j\,0.00064}{4} \\
\overline{Z}_{LD1-0} &= \infty \quad , \quad \text{since } \Delta\text{-connected}
\end{aligned} \tag{8.122}$$

Next, the admittance matrix of the positive-sequence network (i.e. Y_1) will be formed in a manner described in section 8.4. Bus 1 is included in order to determine the voltage at all buses prior to the connection of the unsymmetrical load. However, the load $LD2$ is not included in the Y-bus matrix since it is unsymmetrical. It implies that Y_1 is identical with the admittance matrix Y in Example 6.3 with the exception of the fifth diagonal element in which the unsymmetrical load LD2 is not included, i.e. $Y_1 = Y$ with $\overline{Y}_{55} = \frac{1}{\overline{Z}_{t2-1}}$, see equation (6.49).

2) Next step is to replace the sequence networks with Thévenin equivalents as seen from bus 5, i.e. $\overline{U}_{Thbus5}, \overline{Z}_{Thbus5-1}, \overline{Z}_{Thbus5-2}$ and $\overline{Z}_{Thbus5-0}$, see Figure 8.11.

First, the Z-bus matrix is calculated as follows:

$$\mathbf{Z_1} = \mathbf{Y_1^{-1}} = \begin{bmatrix}
0.6429{+}j0.5264 & 0.6429{+}j0.4827 & 0.6412{+}j0.4797 & 0.6429{+}j0.4827 & 0.6429{+}j0.4827 \\
0.6429{+}j0.4827 & 0.6429{+}j0.4827 & 0.6412{+}j0.4797 & 0.6429{+}j0.4827 & 0.6429{+}j0.4827 \\
0.6412{+}j0.4797 & 0.6412{+}j0.4797 & 0.6412{+}j0.4797 & 0.6412{+}j0.4797 & 0.6412{+}j0.4797 \\
0.6429{+}j0.4827 & 0.6429{+}j0.4827 & 0.6412{+}j0.4797 & 0.6437{+}j0.4842 & 0.6437{+}j0.4842 \\
0.6429{+}j0.4827 & 0.6429{+}j0.4827 & 0.6412{+}j0.4797 & 0.6437{+}j0.4842 & 0.6437{+}j0.6175
\end{bmatrix}$$

Then, based on equation (8.110), the Thévenin voltage $\overline{U}_{Thbus5}$ can be obtained as

$$\overline{U}_{Thbus5} = \mathbf{Z_1}(5,1)\overline{I}_{bus1-1} = \mathbf{Z_1}(5,1)\,\frac{\overline{U}_{Th}}{\mathbf{Z_1}(1,1)} = 0.968\angle -2.413^\circ \tag{8.123}$$

Based on equations (8.113)-(8.114), the positive-sequence impedance of the Thévenin equivalent can be obtained from the Z-bus matrix of the positive-sequence system with voltage source shortened. This implies that in forming $\mathbf{Y_{\Delta 1}}$, the row and column corresponding to the bus connected to voltage source (i.e. bus 1) in matrix $\mathbf{Y_1}$ are removed. Thus,

$$\mathbf{Y_{\Delta 1}} = \begin{bmatrix} \overline{Y}_{22-1} & -\frac{1}{\overline{Z}_{23-1}} & -\frac{1}{\overline{Z}_{24-1}} & 0 \\ -\frac{1}{\overline{Z}_{23-1}} & \overline{Y}_{33-1} & 0 & 0 \\ -\frac{1}{\overline{Z}_{24-1}} & 0 & \overline{Y}_{44-1} & -\frac{1}{\overline{Z}_{t2-1}} \\ 0 & 0 & -\frac{1}{\overline{Z}_{t2-1}} & \frac{1}{\overline{Z}_{t2-1}} \end{bmatrix} \tag{8.124}$$

The matrix $\mathbf{Z_{\Delta 1}}$ can now be obtained as follows:

$$\mathbf{Z_{\Delta 1}} = \mathbf{Y_{\Delta 1}^{-1}} = \begin{bmatrix} 0.0018+j0.0423 & 0.0018+j0.0421 & 0.0018+j0.0423 & 0.0018+j0.0423 \\ 0.0018+j0.0421 & 0.0036+j0.0449 & 0.0018+j0.0421 & 0.0018+j0.0421 \\ 0.0018+j0.0423 & 0.0018+j0.0421 & 0.0026+j0.0438 & 0.0026+j0.0438 \\ 0.0018+j0.0423 & 0.0018+j0.0421 & 0.0026+j0.0438 & 0.0026+j0.1771 \end{bmatrix} \tag{8.125}$$

Note that the element (4,4) corresponds to bus 5 since the row and column corresponding to bus 1 is removed. This implies that

$$\overline{Z}_{Thbus5-1} = 0.0026 + j0.1771 \tag{8.126}$$

The Thévenin impedance of the negative-sequences $\overline{Z}_{Thbus5-2}$ can be calculated using the corresponding matrix of the negative-sequence. The only difference between the positive- and negative-sequence networks is that there is no voltage source in the negative-sequence system.

Since all impedances (and thereby all admittances) in positive- and negative-sequence networks are identical, the following is valid

$$\mathbf{Y_{\Delta 2}} = \mathbf{Y_{\Delta 1}} \tag{8.127}$$

Thus,

$$\overline{Z}_{Thbus5-2} = \overline{Z}_{Thbus5-1} = 0.0026 + j0.1771 \tag{8.128}$$

The Y-bus matrix of the zero-sequence is different compared with the other sequences, both owing to different numerical values but also because of the zero-sequence connections in transformers and loads.

$$\mathbf{Y_{\Delta 0}} = \begin{bmatrix} \overline{Y}_{22-0} & -\frac{1}{\overline{Z}_{23-0}} & -\frac{1}{\overline{Z}_{24-0}} & 0 \\ -\frac{1}{\overline{Z}_{23-0}} & \overline{Y}_{33-0} & 0 & 0 \\ -\frac{1}{\overline{Z}_{24-0}} & 0 & \overline{Y}_{44-0} & -\frac{1}{\overline{Z}_{t2-0}} \\ 0 & 0 & -\frac{1}{\overline{Z}_{t2-0}} & \frac{1}{\overline{Z}_{t2-0}} \end{bmatrix} \tag{8.129}$$

where

$$\begin{aligned} \overline{Y}_{22-0} &= \frac{1}{\overline{Z}_{t1-0}} + \frac{1}{\overline{Z}_{23-0}} + \overline{y}_{sh-23-0} + \frac{1}{\overline{Z}_{24-0}} + \overline{y}_{sh-24-0} \\ \overline{Y}_{33-0} &= \frac{1}{\overline{Z}_{23-0}} + \overline{y}_{sh-23-0} \\ \overline{Y}_{44-0} &= \frac{1}{\overline{Z}_{24-0}} + \overline{y}_{sh-24-0} + \frac{1}{\overline{Z}_{t2-0}} \end{aligned}$$

Corresponding Z-bus matrix is obtained as the inverse

$$\mathbf{Z_{\Delta 0}} = \mathbf{Y_{\Delta 0}^{-1}} = \begin{bmatrix} 0.0000{+}j0.0438 & 0.0000{+}j0.0438 & 0.0000{+}j0.0438 & 0.0000{+}j0.0438 \\ 0.0000{+}j0.0438 & 0.0051{+}j0.0528 & 0.0000{+}j0.0438 & 0.0000{+}j0.0438 \\ 0.0000{+}j0.0438 & 0.0000{+}j0.0438 & 0.0026{+}j0.0483 & 0.0026{+}j0.0483 \\ 0.0000{+}j0.0438 & 0.0000{+}j0.0438 & 0.0026{+}j0.0483 & 0.0026{+}j0.1816 \end{bmatrix} \tag{8.130}$$

Note that element (4,4) corresponds to bus 5. Thus,

$$\overline{Z}_{Thbus5-0} = 0.0026 + j0.1816 \tag{8.131}$$

Next, since all Thévenin equivalents as seen from bus 5 are identified, the voltage vector and impedance matrix expressed in equation (8.116) can now be obtained as follows:

$$\begin{aligned} \mathbf{U_{pre}}(5) &= \begin{bmatrix} \overline{U}_{Thbus5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.968\angle -2.413° \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{Z_{\Delta}}(5,5) &= \begin{bmatrix} \overline{Z}_{Thbus5-1} & 0 & 0 \\ 0 & \overline{Z}_{Thbus5-2} & 0 \\ 0 & 0 & \overline{Z}_{Thbus5-0} \end{bmatrix} = \\ &= \begin{bmatrix} 0.0026 + j0.1771 & 0 & 0 \\ 0 & 0.0026 + j0.1771 & 0 \\ 0 & 0 & 0.0026 + j0.1816 \end{bmatrix} \end{aligned} \tag{8.132}$$

3) Determine the symmetrical components of the unsymmetrical load by using equation (8.118) and $\overline{Z}_{LD2pu}$ from the solution to Example 6.3.

$$\begin{aligned} \mathbf{Z_{LDbus5s}} = \mathbf{T}^{-1}\mathbf{Z_{LDbus5p_h}}\mathbf{T} = \mathbf{T}^{-1} \begin{bmatrix} 2\overline{Z}_{LD2pu} & 0 & 0 \\ 0 & \overline{Z}_{LD2pu} & 0 \\ 0 & 0 & \overline{Z}_{LD2pu} \end{bmatrix} \mathbf{T} = \\ = \begin{bmatrix} 3.0083 + j0.9888 & 0.7521 + j0.2472 & 0.7521 + j0.2472 \\ 0.7521 + j0.2472 & 3.0083 + j0.9888 & 0.7521 + j0.2472 \\ 0.7521 + j0.2472 & 0.7521 + j0.2472 & 3.0083 + j0.9888 \end{bmatrix} \end{aligned} \tag{8.133}$$

The symmetrical components of the currents through the load can now be determined by using equations (8.119) and (8.132).

$$\mathbf{I_{\Delta}}(5) = -\left[\mathbf{Z_{\Delta}}(5,5) + \mathbf{Z_{LDbus5s}}\right]^{-1} \mathbf{U_{pre}}(5) = \begin{bmatrix} 0.3315\angle 155.8442° \\ 0.0653\angle -26.5244° \\ 0.0653\angle -26.6221° \end{bmatrix} = \begin{bmatrix} \mathbf{I_{\Delta 1}}(5) \\ \mathbf{I_{\Delta 2}}(5) \\ \mathbf{I_{\Delta 0}}(5) \end{bmatrix} \tag{8.134}$$

4) The symmetrical components of all voltages can be calculated by using equations (8.110) and (8.120).

$$\begin{aligned}
\mathbf{U_1'}(2) &= \mathbf{Z_1}(2,1)\cdot \overline{I}_{bus1-1} + \mathbf{Z_{\Delta 1}}(2,5)\mathbf{I_{\Delta 1}}(5) = 0.9618\angle-3.1760^\circ \\
\mathbf{U_2'}(2) &= \mathbf{Z_{\Delta 2}}(2,5)\mathbf{I_{\Delta 2}}(5) = 0.0028\angle 61.0623^\circ \\
\mathbf{U_0'}(2) &= \mathbf{Z_{\Delta 0}}(2,5)\mathbf{I_{\Delta 0}}(5) = 0.0029\angle 63.3779^\circ \\
\mathbf{U_1'}(3) &= \mathbf{Z_1}(3,1)\cdot \overline{I}_{bus1-1} + \mathbf{Z_{\Delta 1}}(3,5)\mathbf{I_{\Delta 1}}(5) = 0.9581\angle-3.2745^\circ \\
\mathbf{U_2'}(3) &= \mathbf{Z_{\Delta 2}}(3,5)\mathbf{I_{\Delta 2}}(5) = 0.0028\angle 60.9638^\circ \\
\mathbf{U_0'}(3) &= \mathbf{Z_{\Delta 0}}(3,5)\mathbf{I_{\Delta 0}}(5) = 0.0029\angle 63.3778^\circ \\
\mathbf{U_1'}(4) &= \mathbf{Z_1}(4,1)\cdot \overline{I}_{bus1-1} + \mathbf{Z_{\Delta 1}}(4,5)\mathbf{I_{\Delta 1}}(5) = 0.9614\angle-3.1976^\circ \\
\mathbf{U_2'}(4) &= \mathbf{Z_{\Delta 2}}(4,5)\mathbf{I_{\Delta 2}}(5) = 0.0029\angle 60.0356^\circ \\
\mathbf{U_0'}(4) &= \mathbf{Z_{\Delta 0}}(4,5)\mathbf{I_{\Delta 0}}(5) = 0.0032\angle 60.3527^\circ \\
\mathbf{U_1'}(5) &= \mathbf{Z_1}(5,1)\cdot \overline{I}_{bus1-1} + \mathbf{Z_{\Delta 1}}(5,5)\mathbf{I_{\Delta 1}}(5) = 0.9465\angle-5.6970^\circ \\
\mathbf{U_2'}(5) &= \mathbf{Z_{\Delta 2}}(5,5)\mathbf{I_{\Delta 2}}(5) = 0.0116\angle 62.6242^\circ \\
\mathbf{U_0'}(5) &= \mathbf{Z_{\Delta 0}}(5,5)\mathbf{I_{\Delta 0}}(5) = 0.0119\angle 62.5733^\circ
\end{aligned} \tag{8.135}$$

Note that the element numbers given in the above equations are the bus numbers. However, for $\mathbf{Z_\Delta}$ matrices, since the row and column corresponding to the bus connected to the voltage source (in this example bus 1) are removed, $\mathbf{Z_\Delta}(k,r)$ corresponds to the element $\mathbf{Z_\Delta}(k-1,r-1)$, i.e. with $\mathbf{Z_\Delta}(2,5)$ it means the element $\mathbf{Z_\Delta}(1,4)$.

Since $\bar{Z}_t$ and $\bar{y}_{sh}$ are lossless, the system losses are in the lines, i.e. $\bar{Z}_{23}$ and $\bar{Z}_{24}$. The positive-, negative- and zero-sequence currents through these impedances are expressed by

$$\begin{aligned}
\overline{I}_{Z23-1} &= \frac{\mathbf{U_1'}(2)-\mathbf{U_1'}(3)}{\overline{Z}_{23-1}} = 1.1972\angle-401209^\circ \\
\overline{I}_{Z23-2} &= \frac{\mathbf{U_2'}(2)-\mathbf{U_2'}(3)}{\overline{Z}_{23-2}} = 0.0034\angle-24.1174^\circ \\
\overline{I}_{Z23-0} &= \frac{\mathbf{U_0'}(2)-\mathbf{U_0'}(3)}{\overline{Z}_{23-0}} = 0.0000\angle 153.3778^\circ \\
\overline{I}_{Z24-1} &= \frac{\mathbf{U_1'}(2)-\mathbf{U_1'}(4)}{\overline{Z}_{24-1}} = 0.3314\angle-24.1061^\circ \\
\overline{I}_{Z24-2} &= \frac{\mathbf{U_2'}(2)-\mathbf{U_2'}(4)}{\overline{Z}_{24-2}} = 0.0653\angle 153.4756^\circ \\
\overline{I}_{Z24-0} &= \frac{\mathbf{U_0'}(2)-\mathbf{U_0'}(4)}{\overline{Z}_{24-0}} = 0.0653\angle 153.3779^\circ
\end{aligned} \tag{8.136}$$

Positive-, negative- and zero-sequence powers injected into the line with impdance $\bar{Z}_{23}$ are given by

$$\begin{aligned}
\overline{S}_{Z23-1} &= \mathbf{U_1'}(2)\overline{I}^*_{Z23-1} = 0.9203 + j0.6921 \\
\overline{S}_{Z23-2} &= \mathbf{U_2'}(2)\overline{I}^*_{Z23-2} = (7.60 + j5.72)\times 10^{-6} \\
\overline{S}_{Z23-0} &= \mathbf{U_0'}(2)\overline{I}^*_{Z23-0} = 4.24\times 10^{-15} + j2.61\times 10^{-15}
\end{aligned} \tag{8.137}$$

By using equation (8.25), the total power flowing through $\bar{Z}_{23}$ is given by

$$\overline{S}_{Z23_{in}} = \overline{S}_{Z23-1} + \overline{S}_{Z23-2} + \overline{S}_{Z23-0} = 0.4602 + j0.3461 \tag{8.138}$$

In a similar manner, the following can also be obtained:

$$\begin{aligned}
\overline{S}_{Z24_{in}} &= \mathbf{U_1'}(2)\overline{I}^*_{Z24-1} + \mathbf{U_2'}(2)\overline{I}^*_{Z24-2} + \mathbf{U_0'}(2)\overline{I}^*_{Z24-0} = 0.1489 + j0.0567 \\
\overline{S}_{Z23_{out}} &= \mathbf{U_1'}(3)\overline{I}^*_{Z23-1} + \mathbf{U_2'}(3)\overline{I}^*_{Z23-2} + \mathbf{U_0'}(3)\overline{I}^*_{Z23-0} = 0.4589 + j0.3461 \\
\overline{S}_{Z24_{out}} &= \mathbf{U_1'}(4)\overline{I}^*_{Z24-1} + \mathbf{U_2'}(4)\overline{I}^*_{Z24-2} + \mathbf{U_0'}(4)\overline{I}^*_{Z24-0} = 0.1488 + j0.0567
\end{aligned}$$

The efficiency can now be obtained as:

$$\eta = 100 \cdot \frac{Real(\overline{S}_{Z23_{out}}) + Real(\overline{S}_{Z24_{out}})}{Real(\overline{S}_{Z23_{in}}) + Real(\overline{S}_{Z24_{in}})} = 99.7911\% \tag{8.139}$$

The efficiency can also be calculated as follows:

$$\overline{S}_{inj} = \mathbf{U_1'}(2)\left(\frac{\overline{U}_{Th} - \mathbf{U_1'}(2)}{\overline{Z}_{t1-1}}\right)^* + \mathbf{U_2'}(2)\left(\frac{0 - \mathbf{U_2'}(2)}{\overline{Z}_{t1-2}}\right)^* + \mathbf{U_0'}(2)\left(\frac{0 - \mathbf{U_0'}(2)}{\overline{Z}_{t1-0}}\right)^*$$

where, $\overline{S}_{inj}$ is the total power injected into the system by the infinite bus. Next, the total load is calculated as follows:

$$\overline{S}_{LD_{tot}} = \frac{|\mathbf{U_1'}(3)|^2}{\overline{Z}^*_{LD1-1}} + \frac{|\mathbf{U_2'}(3)|^2}{\overline{Z}^*_{LD1-2}} + 0 + \mathbf{U_1'}(5)[-\mathbf{I_{\Delta 1}}(5)]^* + \mathbf{U_2'}(5)[-\mathbf{I_{\Delta 2}}(5)]^* + \mathbf{U_0'}(5)[-\mathbf{I_{\Delta 0}}(5)]^*$$

Thus,

$$\eta = 100 \cdot \frac{Real(\overline{S}_{LD_{tot}})}{Real(\overline{S}_{inj})}\,\% \tag{8.140}$$

Note that (8.139) is valid if the losses are only in the lines. However, (8.140) is a general expression regardless of where the losses are.

Appendix A

Matlab-codes for Examples in Chapters 6-7

A.1 Example 6.2

```
clear

deg=180/pi;
rad=1/deg;

%--- Example 6.2
% Choose the base values
Sb=0.5; Ub10=10; Ib10=Sb/Ub10/sqrt(3);Zb10=Ub10^2/Sb;
Ub70=70; Ib70=Sb/Ub70/sqrt(3);

%Calculate the per-unit values of the Thevenin equivalent of the system
UTh=70*exp(j*0*rad);
Isc=0.3*exp(j*-90*rad);
UThpu =UTh/Ub70;
Iscpu =Isc/Ib70;
ZThpu =UThpu/Iscpu;

% Calculate the per-unit values of the transformer
Zt=j*4/100;Snt=5;
Ztpu=Zt*Sb/Snt;

%Calculate the per-unit values of the line
Z21pu=5*(0.9+j*0.3)/Zb10;
ysh21pu=5*(j*3*1E-6)*Zb10/2;

%Calculate the per-unit values of the industry impedance
cosphi=0.8;sinphi=sqrt(1-cosphi^2);
Un=Ub10;PLD=0.4;absSLD=PLD/cosphi;
SLD=absSLD*(cosphi+j*sinphi);
ZLDpu=Un^2/conj(SLD)/Zb10

% The twoport of the system
AL=1+ysh21pu*Z21pu;
BL=Z21pu;
CL=ysh21pu*(2+ysh21pu*Z21pu);
DL=AL;

F_L=[AL BL ; CL DL];
```

```
F_Th_tr=[1 ZThpu+Ztpu ; 0 1];
F_tot=F_Th_tr*F_L;

%The impedance of the entire system
Ztotpu=(F_tot(1,1)*ZLDpu+F_tot(1,2))/(F_tot(2,1)*ZLDpu+F_tot(2,2))
I4pu = UThpu/Ztotpu;

%The power fed by the transformer into the line
U2pu_I2pu=inv(F_Th_tr)*[UThpu;I4pu];
S2=U2pu_I2pu(1,1)*conj(U2pu_I2pu(2,1))*Sb

%The voltage at the industry
U1pu_I1pu=inv(F_tot)*[UThpu;I4pu];
U1=abs(U1pu_I1pu(1,1))*Ub10,
```

A.2 Example 6.3

```
clear

deg=180/pi;
rad=1/deg;

%--- Example 6.3
% Choose the base values
Sb=0.5; Ub70=70; Ib70=Sb/Ub70/sqrt(3);
Ub10=10; Ib10=Sb/Ub10/sqrt(3);Zb10=Ub10^2/Sb;
Ub04=04; Ib04=Sb/Ub04/sqrt(3);Zb04=Ub04^2/Sb;

%Calculate the per-unit values of the inÂ¯nite bus
U1=70/Ub70;
%Calculate the per-unit values of the transformer T1 and T2
Zt1=j*7/100;Snt1=0.8;
Zt1pu=Zt1*Sb/Snt1;

Zt2=j*8/100;Snt2=0.3;
Zt2pu=Zt2*Sb/Snt2;

%Calculate the per-unit values of Line1and Line2
Z23pu=2*[0.17+j*0.3]/Zb10;
ysh23pu=2*(j*3.2*1E-6)*Zb10/2;

Z24pu=1*[0.17+j*0.3]/Zb10;
ysh24pu=1*(j*3.2*1E-6)*Zb10/2;
```

```
%Calculate the per-unit values of the impedance LD1 and LD2
cosphiLD1=0.8;sinphiLD1=sqrt(1-cosphiLD1^2);
UnLD1=Ub10;PLD1=0.5;absSLD1=PLD1/cosphiLD1;
SLD1=absSLD1*(cosphiLD1+j*sinphiLD1);
ZLD1pu=UnLD1^2/conj(SLD1)/Zb10;

cosphiLD2=0.95;sinphiLD2=sqrt(1-cosphiLD2^2);
UnLD2=Ub04;PLD2=0.2;absSLD2=PLD2/cosphiLD2;
SLD2=absSLD2*(cosphiLD2+j*sinphiLD2);
ZLD2pu=UnLD2^2/conj(SLD2)/Zb04;

%Y-BUS
Y22=1/Zt1pu+1/Z23pu+ysh23pu+1/Z24pu+ysh24pu;
Y33=1/Z23pu+ysh23pu+1/ZLD1pu;
Y44=1/Z24pu+ysh24pu+1/Zt2pu;

Ybus=[ 1/Zt1pu   -1/Zt1pu      0           0             0;
       -1/Zt1pu     Y22      -1/Z23pu   -1/Z24pu         0;
          0       -1/Z23pu     Y33         0             0;
          0       -1/Z24pu      0          Y44       -1/Zt2pu;
          0          0          0       -1/Zt2pu  1/Zt2pu+1/ZLD2pu];
Zbus=inv(Ybus);

%Calculate the efficiency
I1=U1/Zbus(1,1);
U2=Zbus(2,1)*I1;
U3=Zbus(3,1)*I1;
U4=Zbus(4,1)*I1;
U5=Zbus(5,1)*I1;

S1=U1*conj(I1)*Sb;
IZ23=(U2-U3)/Z23pu;
IZ24=(U2-U4)/Z24pu;
PfLine1=real(Z23pu)*abs(IZ23)^2*Sb;
PfLine2=real(Z24pu)*abs(IZ24)^2*Sb;

eta=(real(S1)-PfLine1-PfLine2)/real(S1);

Zf4=0;
YD=Ybus(2:5,2:5);
ZD=inv(YD);
Isc4=U4*Ib10/(Zf4+ZD(4-1,4-1));
absIsc4=abs(Isc4);
angIsc4=angle(Isc4)*deg;
```

A.3 Example 7.10

```
% Start of file
clear,
clear global

deg=180/pi;
maxiter=10;
EPS=1e-4;

k1=-0.2; k2=1.2; k3=-0.07; k4=0.4;

% Step 1
converged=0; iter=0; x=3/deg;

while ~converged & iter < maxiter,

% Step 2
delta_gx=k4-(k1*x+k2*cos(x-k3));

% Step Final
    if all(abs(delta_gx)< EPS),
        converged=1;
        iter=iter,
        xdeg=x*deg
    else
 % Step 3
        Jac=k1-k2*sin(x-k3); %Jac=dfx/dx;
 % Step 4
        delta_x=inv(Jac)*delta_gx;
 % Step 5
        x=x+delta_x;
        iter=iter+1;
    end, % if all

    if iter==maxiter,
        iter=iter,
        disp('The equation has no solutions')
        disp('or')
        disp('bad initial value, try with another initial value')
    end, % iter
end, % while
% End of file
```

A.4 Example 7.12

```
% Start of file
clear,%
clear global

Sbase=100; Ubase=220; deg=180/pi;
%Step 1
% 1a)
U1=1;theta1=0;PLD1=0.2;QLD1=0.02;%
U2=1;PG2=1;PLD2=2;QLD2=0.2;
%1b)
Z12=0.02+j*0.2;%
Y=[1/Z12 -1/Z12 ; -1/Z12 1/Z12];%
G=real(Y); B=imag(Y);%
PGD2=PG2-PLD2;
%1c)
theta2=0;

iter=0;%
while iter < 3,
   iter=iter+1;
   %Step 2
   %2a)
   P2=U1*U2*(G(2,1)*cos(theta2-theta1)+B(2,1)*sin(theta2-theta1))+U2^2*G(2,2);
   %2b
   deltaP=PGD2-P2;
   %Step 3
   Q2=U2*U1*(G(2,1)*sin(theta2-theta1)-B(2,1)*cos(theta2-theta1))-U2^2*B(2,2);
   H=-Q2-U2^2*B(2,2);
   JAC=[H];
   %Step 4
   DX=inv(JAC)*[deltaP];
   delta_theta2=DX;
   %Step 5
   theta2=theta2+delta_theta2;
end,

%Step final

P1=U1*U2*(G(1,2)*cos(theta1-theta2)+B(1,2)*sin(theta1-theta2))+U1^2*G(1,1);%
Q1=U1*U2*(G(1,2)*sin(theta1-theta2)-B(1,2)*cos(theta1-theta2))-U1^2*B(1,1);
Q2=U2*U1*(G(2,1)*sin(theta2-theta1)-B(2,1)*cos(theta2-theta1))-U2^2*B(2,2);
PG1=(P1+PLD1)*Sbase; QG1=(Q1+QLD1)*Sbase; QG2=(Q2+QLD2)*Sbase;

g=-G;b=-B;bsh_12=0;
```

```
P_12=(U1^2*g(1,2)-U1*U2*(g(1,2)*cos(theta1-theta2)+b(1,2)*sin(theta1-theta2)))*Sbase;
P_21=(U2^2*g(2,1)-U2*U1*(g(2,1)*cos(theta2-theta1)+b(2,1)*sin(theta2-theta1)))*Sbase;
Q_12=((-bsh_12-b(1,2))*U1^2-...
     U1*U2*(g(1,2)*sin(theta1-theta2)-b(1,2)*cos(theta1-theta2)))*Sbase;
Q_21=((-bsh_12-b(2,1))*U2^2-...
     U2*U1*(g(2,1)*sin(theta2-theta1)-b(2,1)*cos(theta2-theta1)))*Sbase;

PLoss=P_12+P_21; %or PLoss=(PG1+PG2)-(PLD1+PLD2)*Sbase;

ANG=[theta1 theta2]'*deg;
VOLT=[U1 U2]'*Ubase;

% End of file
```

A.5 Example 7.13

```
% Start of file
clear,%
clear global

tole=1e-6;
Sbase=100; Ubase=220; deg=180/pi;
%Step 1
% 1a)
U1=1;theta1=0;PLD1=0.2;QLD1=0.02;
PG2=1;QG2=0.405255;PLD2=2;QLD2=0.2;
%1b)
Z12=0.02+j*0.2;
Y=[1/Z12 -1/Z12 ; -1/Z12 1/Z12];
G=real(Y); B=imag(Y);
PGD2=PG2-PLD2;
QGD2=QG2-QLD2;
%1c)
theta2=0;
U2=1;

P2=U1*U2*(G(2,1)*cos(theta2-theta1)+B(2,1)*sin(theta2-theta1))+U2^2*G(2,2);
Q2=U2*U1*(G(2,1)*sin(theta2-theta1)-B(2,1)*cos(theta2-theta1))-U2^2*B(2,2);
deltaP=PGD2-P2;
deltaQ=QGD2-Q2;

%Step 3
while all(abs([deltaP;deltaQ])> tole),
   H=-Q2-B(2,2)*U2^2;
```

```
   N=P2+G(2,2)*U2^2;
   J=P2-G(2,2)*U2^2;
   L=Q2-B(2,2)*U2^2;
   JAC=[H N ; J L];
   %Step 4
   DX=inv(JAC)*[deltaP;deltaQ];
   %Step 5
   theta2=theta2+DX(1); %DX(1)=delta_theta2
   U2=U2*(1+DX(2));     %DX(2)=delta_U2/U2
   %Step 2
   P2=U1*U2*(G(2,1)*cos(theta2-theta1)+B(2,1)*sin(theta2-theta1))+U2^2*G(2,2);
   Q2=U2*U1*(G(2,1)*sin(theta2-theta1)-B(2,1)*cos(theta2-theta1))-U2^2*B(2,2);
   deltaP=PGD2-P2;
   deltaQ=QGD2-Q2;
end, %while

%Step final

P1=U1*U2*(G(1,2)*cos(theta1-theta2)+B(1,2)*sin(theta1-theta2))+U1^2*G(1,1);%
Q1=U1*U2*(G(1,2)*sin(theta1-theta2)-B(1,2)*cos(theta1-theta2))-U1^2*B(1,1);
Q2=U2*U1*(G(2,1)*sin(theta2-theta1)-B(2,1)*cos(theta2-theta1))-U2^2*B(2,2);
PG1=(P1+PLD1)*Sbase; QG1=(Q1+QLD1)*Sbase; QG2=(Q2+QLD2)*Sbase;

g=-G;b=-B;bsh_12=0;
P_12=(U1^2*g(1,2)-U1*U2*(g(1,2)*cos(theta1-theta2)+b(1,2)*sin(theta1-theta2)))*Sbase;
P_21=(U2^2*g(2,1)-U2*U1*(g(2,1)*cos(theta2-theta1)+b(2,1)*sin(theta2-theta1)))*Sbase;
Q_12=((-bsh_12-b(1,2))*U1^2-...
     U1*U2*(g(1,2)*sin(theta1-theta2)-b(1,2)*cos(theta1-theta2)))*Sbase;
Q_21=((-bsh_12-b(2,1))*U2^2-...
     U2*U1*(g(2,1)*sin(theta2-theta1)-b(2,1)*cos(theta2-theta1)))*Sbase;

PLoss=P_12+P_21; %or PLoss=(PG1+PG2)-(PLD1+PLD2)*Sbase;

ANG=[theta1 theta2]'*deg;
VOLT=[U1 U2]'*Ubase;

% End of file
```

A.6 Example 7.14

```
% To run Load Flow (LF), two MATLAB-files are used, namely
% (run_LF.m) and (solve_lf.m)

% Start of file (run_LF.m)
```

```
clear%
clear global
%%%%%%%%%%%%%%%%%%%%%%%%%%
tole=1e-9; deg=180/pi; rad=1/deg ;
%%%%%%%%%%%%%%
% Base values
%%%%%%%%%%%%%%
Sbase=100; Ubase=220; Zb=Ubase^2/Sbase;
%%%%%%%%%%%%
% Bus data
%%%%%%%%%%%%
% Number of buses
nbus=4;
%Bus 1, slack bus
U1=220/Ubase; theta1=0*rad; PLD1=10/Sbase; QLD1=2/Sbase;
%Bus 2, PQ-bus
PG2=0/Sbase; QG2=0/Sbase; PLD2=90/Sbase; QLD2=10/Sbase;
%Bus 3, PQ-bus
PG3=0/Sbase; QG3=0/Sbase; PLD3=80/Sbase; QLD3=10/Sbase;
%Bus 4, PQ-bus
PG4=0/Sbase; QG4=0/Sbase; PLD4=50/Sbase; QLD4=10/Sbase;
%%%%%%%%%%%%
% Line data
%%%%%%%%%%%%
Z12=(5+j*65)/Zb;bsh12=0.0002*Zb;%
Z13=(4+j*60)/Zb;bsh13=0.0002*Zb;%
Z23=(5+j*68)/Zb;bsh23=0.0002*Zb;%
Z34=(3+j*30)/Zb;bsh34=0;
%%%%%%%%%%%%%%
% YBUS matrix
%%%%%%%%%%%%%%
y11=1/Z12+1/Z13+j*bsh12+j*bsh13; y12=-1/Z12; y13=-1/Z13; y14=0;%
y21=-1/Z12; y22=1/Z12+1/Z23+j*bsh12+j*bsh23; y23=-1/Z23; y24=0;%
y31=-1/Z13; y32=-1/Z23; y33=1/Z13+1/Z23+1/Z34+j*bsh13+j*bsh23; y34=-1/Z34;%
y41=0; y42=0; y43=-1/Z34; y44=1/Z34;%
YBUS=[ ];% Define YBUS
G=real(YBUS); B=imag(YBUS);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PGD for PU- and PQ-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
PGD2=PG2-PLD2; % for bus 2 (PQ-bus)
PGD3=PG3-PLD3; % for bus 3 (PQ-bus)
PGD4=PG4-PLD4; % for bus 4 (PQ-bus)
PGD=[PGD2 ; PGD3 ; PGD4];
%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Define the column vector QGD for PQ-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%

QGD=[ ];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Use fsolve function in MATLAB to run load flow
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Unknown variables [theta2 theta3 theta4 U2 U3 U4]';
% Define the initial values of the unknown variables
X0=[0 0 0 1 1 1]'; % Flat initial values
s_z=size(X0);
nx=s_z(1,1); % number of unknown variables
% The function below is used for fsolve (type "help fsolve" in MATLAB)
options_solve=optimset('Display','off','TolX',tole,'TolFun',tole);

% Parameters used for fsolve
PAR=[nx ; nbus ; U1 ; theta1];% U1 and theta1 are known (slack bus).
[X_X,FVAL,EXITFLAG,OUTPUT]=fsolve('solve_lf',X0,options_solve,G,B,PGD,QGD,PAR);

if EXITFLAG~=1,
    disp('No solution'),
    EXITFLAG=EXITFLAG,
    return
end,


% Solved variables X_X=[theta2 theta3 theta4 U2 U3 U4]';
ANG=[theta1 X_X(1) X_X(2) X_X(3)]';% Voltage phase angles
VOLT=[U1 X_X(4) X_X(5) X_X(6)]'; % Voltage magnitudes

ANG_deg=ANG*deg;% in degrees
VOLT_kV=VOLT*Ubase; % in kV

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The generated active and reactive power at slack bus and PU-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
g=-G;b=-B;
% At slack bus, bus 1
P12=g(1,2)*VOLT(1)^2-...
    VOLT(1)*VOLT(2)*(g(1,2)*cos(ANG(1)-ANG(2))+b(1,2)*sin(ANG(1)-ANG(2)));
P13=g(1,3)*VOLT(1)^2-...
    VOLT(1)*VOLT(3)*(g(1,3)*cos(ANG(1)-ANG(3))+b(1,3)*sin(ANG(1)-ANG(3)));
Q12=(-bsh12-b(1,2))*VOLT(1)^2-...
    VOLT(1)*VOLT(2)*(g(1,2)*sin(ANG(1)-ANG(2))-b(1,2)*cos(ANG(1)-ANG(2)));
Q13=(-bsh13-b(1,3))*VOLT(1)^2-...
    VOLT(1)*VOLT(3)*(g(1,3)*sin(ANG(1)-ANG(3))-b(1,3)*cos(ANG(1)-ANG(3)));%
```

```
PG1=P12+P13+PLD1;%
QG1=Q12+Q13+QLD1;%
% You may also use equation (7.43) (Pk and Qk) to find PG1 and QG1
PG1= ; % based on equation (7.43)
QG1= ; % based on equation (7.43)

PG1_MW=PG1*Sbase;%
QG1_MVAr=QG1*Sbase;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%
% Losses
%%%%%%%%%
PLoss_tot=(PG1+PG2+PG3+PG4)-(PLD1+PLD2+PLD3+PLD4);%
PLoss_tot_MW=PLoss_tot*Sbase;


PLoss_Sys1=  ;% Find the power losses in System 1 (pu)
PLoss_Sys1_MW=PLoss_Sys1*Sbase;

PLoss_Sys2_MW= ; % Find the power losses in System 2 (MW)

% End of file (run_LF.m)

%%%%%%%%%%%%%
% Second file
%%%%%%%%%%%%%
% Start of file (solve_lf.m)
% This function solves g(x)=0 for x.

function [g_x]=solve_lf(X,G,B,PGD,QGD,PAR);

nx=PAR(1); nbus=PAR(2); U1=PAR(3); theta1=PAR(4);

PGD2=PGD(1); PGD3=PGD(2); PGD4=PGD(3); QGD2=QGD(1); QGD3=QGD(2); QGD4=QGD(3);

theta2=X(1); theta3=X(2); theta4=X(3); U2=X(4); U3=X(5); U4=X(6);

ANG=[theta1 theta2 theta3 theta4]' ;  VOLT=[U1 U2 U3 U4]';

% We have nx unknown variables, therefore the
% size of g(x) is nx by 1.

g_x=zeros(nx,1);
```

```
%Based on equation (7.43), find Pk and Qk
P2=  ;
P3=  ;
P4=  ;
Q2=  ;
Q3=  ;
Q4=  ;

% Active power mismatch (PU- and PQ-buses)
% Bus 2
g_x(1)=P2-PGD2;
% Bus 3
g_x(2)=P3-PGD3;
% Bus 4
g_x(3)=P4-PGD4;
% Reactive power mismatch (PQ-buses)
% Bus 2
g_x(4)=Q2-QGD2;
% Bus 3
g_x(5)=Q3-QGD3;
% Bus 4
g_x(6)=Q4-QGD4;

% End of file (solve_lf.m)
```

A.7 Example 7.15

```
% Changes in file run_LF.m

%%%%%%%%%%%%
% Bus data
%%%%%%%%%%%%
% Number of buses
nbus=4;

%Bus 3, PU-bus
PG3=0/Sbase; U3=220/Ubase; PLD3=80/Sbase; QLD3=10/Sbase;

%%%%%%%%%%%%
% Line data
%%%%%%%%%%%%

%%%%%%%%%%%%%%
```

```
% YBUS matrix
%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Define PGD for PU- and PQ-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%
% Define QGD for PQ-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%
QGD2=QG2-QLD2; % for bus 2 (PQ-bus)
QGD4=QG4-QLD4; % for bus 4 (PQ-bus)
QGD=[QGD2 ; QGD4];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Use fsolve function in MATLAB to run load flow
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Unknown variables [theta2 theta3 theta4 U2 U4]';
% Define the initial values of the unknown variables
X0=[0 0 0 1 1]'; % Flat initial values

% Parameters used for fsolve
PAR=[nx ; nbus ; U1 ; U3 ; theta1];%

% Solved variables X_X=[theta1 theta2 theta4 U2 U4]';
ANG=[theta1 X_X(1) X_X(2) X_X(3)]';%
VOLT=[U1 X_X(4) U3 X_X(5)]';


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The generated active and reactive power at slack bus and PU-buses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
g=-G;b=-B;
% At slack bus, bus 1





% At PU-buses, bus 3

QG3=  ;%
QG3_MVAr=QG3*Sbase;

%%%%%%%%%
% Losses
%%%%%%%%%
```

```
% End of file (run_LF.m)


%%%%%%%%%%%%%%
% Second file
%%%%%%%%%%%%%%
% Start of file (solve_lf.m)

function [g_x]=solve_lf(X,G,B,PGD,QGD,PAR);

nx=PAR(1); nbus=PAR(2); U1=PAR(3); U3=PAR(4); theta1=PAR(5);

PGD2=PGD(1); PGD3=PGD(2); PGD4=PGD(3); QGD2=QGD(1); QGD4=QGD(2);

theta2=X(1); theta3=X(2); theta4=X(3); U2=X(4); U4=X(5);

ANG=[theta1 theta2 theta3 theta4]'; VOLT=[U1 U2 U3 U4]';

% We have nx unknown variables, therefore the
% size of g(x) is nx by 1.

g_x=zeros(nx,1);

 for m=1:nbus
  for n=1:nbus
   PP(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*cos(ANG(m)-ANG(n))+B(m,n)*sin(ANG(m)-ANG(n)));
   QQ(m,n)=VOLT(m)*VOLT(n)*(G(m,n)*sin(ANG(m)-ANG(n))-B(m,n)*cos(ANG(m)-ANG(n)));
  end, %for n
 end, % for m
 P=sum(PP')';
 Q=sum(QQ')';

% Active power mismatch (PU- and PQ-buses)
% Bus 2
g_x(1)=P(2)-PGD2;
% Bus 3
g_x(2)=P(3)-PGD3;
% Bus 4
g_x(3)=P(4)-PGD4;
% Reactive power mismatch (PQ-buses)
% Bus 2
g_x(4)=Q(2)-QGD2;
% Bus 4
g_x(5)=Q(4)-QGD4;

% End of file (solve_lf.m)
```

A.8 Example 8.5

```
clear

% 1) Start with the building of the impedance diagram of the positive-, negative-
% and zero-sequence for the whole system excluding the unsymmetrical load

Ex6_2, % Run Example 6.2

%Positive- and negative-sequence components in per-unit values (from the
%solution to Example 6.2):

UTh=UThpu;
ZTh_1=ZThpu;
ZTh_2=ZTh_1;
Zt_1=Ztpu;
Z21_1=Z21pu;
Z21_2=Z21_1;
ysh21_1=ysh21pu;
ysh21_2=ysh21_1;

AL_1=AL ; AL_2=AL_1;
BL_1=BL ; BL_2=BL_1;
CL_1=CL ; CL_2=CL_1;
DL_1=DL ; DL_2=DL_1;

A_1=F_tot(1,1) ; A_2=A_1;
B_1=F_tot(1,2) ; B_2=B_1;
C_1=F_tot(2,1) ; C_2=C_1;
D_1=F_tot(2,2) ; D_2=D_1;

%Zero-sequence components in per-unit values
Isc1phi=0.2*exp(-j*90*rad)/Ib70;
ZTh_0=3*UTh/Isc1phi-2*ZTh_1;
Zt_0=Zt_1;
Z21_0=3*Z21_1;
ysh21_0=0.5*ysh21_1;

AL_0=1+ysh21_0*Z21_0;
BL_0=Z21_0;
CL_0=ysh21_0*(2+ysh21_0*Z21_0);
DL_0=AL_0;

% 2) Next step is to replace the networks with Thevenin equivalents as seen
% from the industry connection point (bus 1)
```

```
UThbus1=UTh/A_1;%
ZThbus1_1=B_1/A_1;
ZThbus1_2=ZThbus1_1;

F_tot_0=[ 1  Zt_0 ; 0 1]*[AL_0 BL_0 ; CL_0 DL_0];
ZThbus1_0=F_tot_0(1,2)/F_tot_0(1,1);
% 3) Symmetrical components
alfa=exp(j*120*rad);
TT=[1   1  1 ; alfa^2   alfa  1 ; alfa   alfa^2   1];
ZLDs=inv(TT)*[2*ZLDpu 0  0 ; 0   ZLDpu  0 ; 0   0   ZLDpu]*TT;
UTH=[UThbus1 ; 0 ; 0];
Zs=[ZThbus1_1    0     0 ; 0  ZThbus1_2  0 ; 0   0   ZThbus1_0];
Is=inv(Zs+ZLDs)*UTH;

% 4-5) The symmetrical components of the voltage at the industry (bus 1)
Ubus1s=ZLDs*Is;
Ubus2_Ibus2_1=[AL_1 BL_1 ; CL_1 DL_1]*[Ubus1s(1) ; Is(1)];
Ubus2_Ibus2_2=[AL_2 BL_2 ; CL_2 DL_2]*[Ubus1s(2) ; Is(2)];
Ubus2_Ibus2_0=[AL_0 BL_0 ; CL_0 DL_0]*[Ubus1s(3) ; Is(3)];
Sbus2_1=Ubus2_Ibus2_1(1,1)*conj(Ubus2_Ibus2_1(2,1))*Sb;
Sbus2_2=Ubus2_Ibus2_2(1,1)*conj(Ubus2_Ibus2_2(2,1))*Sb;
Sbus2_0=Ubus2_Ibus2_0(1,1)*conj(Ubus2_Ibus2_0(2,1))*Sb;

% 6)
Ubus1_Ph=[abs(TT*Ubus1s*Ub10/sqrt(3)) angle(TT*Ubus1s*Ub10/sqrt(3))*deg]
Stot=Sbus2_1+Sbus2_2+Sbus2_0
```

A.9 Example 8.6

```
clear,

Ex6_3,

% 1)
% Positive- and negative-sequence components in per-unit values (from the
% solution to Example 6.3)
UTh=U1;
Zt1_1=Zt1pu ; Zt1_2=Zt1_1;%
Zt2_1=Zt2pu ; Zt2_2=Zt2_1;%
Z23_1=Z23pu ; Z23_2=Z23_1;%
ysh23_1=ysh23pu ; ysh23_2=ysh23_1;%
Z24_1=Z24pu ; Z24_2=Z24_1;%
ysh24_1=ysh24pu ; ysh24_2=ysh24_1;%
ZLD1_1=ZLD1pu ; ZLD1_2=ZLD1_1;%
```

```
%Zero-sequence components in per-unit values
Zt1_0=Zt1_1 ; Zt2_0=Zt2_1;%
Z23_0=3*Z23_1 ; ysh23_0=0.5*ysh23_1;%
Z24_0=3*Z24_1 ; ysh24_0=0.5*ysh24_1;%
% 2)
Y1=Ybus; Y1(5,5)=1/Zt2_1; Z1=inv(Y1);
Ibus1_1=UTh/Z1(1,1);
UThbus5=Z1(5,1)*Ibus1_1;
YD1=Y1(2:5,2:5);
ZD1=inv(YD1);
ZD2=ZD1;
% 3)
ZThbus5_1=ZD1(5-1,5-1); ZThbus5_2=ZThbus5_1;

Y22_0=1/Zt1_0+1/Z23_0+ysh23_0+1/Z24_0+ysh24_0;
Y33_0=1/Z23_0+ysh23_0;
Y44_0=1/Z24_0+ysh24_0+1/Zt2_0;

YD0=[   Y22_0  -1/Z23_0   -1/Z24_0       0;
     -1/Z23_0     Y33_0       0          0;
     -1/Z24_0      0          Y44_0   -1/Zt2_0;
         0         0      -1/Zt2_0    1/Zt2_0];

ZD0=inv(YD0);
ZThbus5_0=ZD0(5-1,5-1);
UPre_5=[UThbus5 ; 0 ; 0];
ZD_5=[ZThbus5_1    0     0 ; 0  ZThbus5_2  0 ; 0   0   ZThbus5_0];
alfa=exp(j*120*rad);
TT=[1   1  1 ; alfa^2   alfa  1 ; alfa   alfa^2   1];
ZLDbus5s=inv(TT)*[2*ZLD2pu 0  0 ; 0   ZLD2pu  0 ; 0   0   ZLD2pu]*TT;
ID_5=-inv(ZD_5+ZLDbus5s)*UPre_5;

% 4)
Up2_1=Z1(2,1)*Ibus1_1+ ZD1(2-1,5-1)*ID_5(1);
Up2_2=      0        + ZD2(2-1,5-1)*ID_5(2);
Up2_0=      0        + ZD0(2-1,5-1)*ID_5(3);
Up3_1=Z1(3,1)*Ibus1_1+ ZD1(3-1,5-1)*ID_5(1);
Up3_2=      0        + ZD2(3-1,5-1)*ID_5(2);
Up3_0=      0        + ZD0(3-1,5-1)*ID_5(3);
Up4_1=Z1(4,1)*Ibus1_1+ ZD1(4-1,5-1)*ID_5(1);
Up4_2=      0        + ZD2(4-1,5-1)*ID_5(2);
Up4_0=      0        + ZD0(4-1,5-1)*ID_5(3);
Up5_1=Z1(5,1)*Ibus1_1+ ZD1(5-1,5-1)*ID_5(1);
Up5_2=      0        + ZD2(5-1,5-1)*ID_5(2);
Up5_0=      0        + ZD0(5-1,5-1)*ID_5(3);
```

```
IZ23_1=(Up2_1-Up3_1)/Z23_1;
IZ23_2=(Up2_2-Up3_2)/Z23_2;
IZ23_0=(Up2_0-Up3_0)/Z23_0;
IZ24_1=(Up2_1-Up4_1)/Z24_1;
IZ24_2=(Up2_2-Up4_2)/Z24_2;
IZ24_0=(Up2_0-Up4_0)/Z24_0;

SZ23_1=Up2_1*conj(IZ23_1);
SZ23_2=Up2_2*conj(IZ23_2);
SZ23_0=Up2_0*conj(IZ23_0);
SZ23_in=SZ23_1+SZ23_2+SZ23_0;
SZ24_in=Up2_1*conj(IZ24_1)+Up2_2*conj(IZ24_2)+Up2_0*conj(IZ24_0);

SZ23_out=Up3_1*conj(IZ23_1)+Up3_2*conj(IZ23_2)+Up3_0*conj(IZ23_0);
SZ24_out=Up4_1*conj(IZ24_1)+Up4_2*conj(IZ24_2)+Up4_0*conj(IZ24_0);

eta=100*(real(SZ23_out)+real(SZ24_out))/(real(SZ23_in)+real(SZ24_in))

Sinj=UTh*conj((UTh-Up2_1)/Zt1_1)+Up2_2*conj((0-Up2_2)/Zt1_2)+...
     Up2_0*conj((0-Up2_0)/Zt1_0);
SLDtot=abs(Up3_1)^2/ZLD1_1+abs(Up3_2)^2/ZLD1_2+0-...
           Up5_1*conj(ID_5(1))-Up5_2*conj(ID_5(2))-Up5_0*conj(ID_5(3));

eta=100*real(SLDtot)/real(Sinj)
```

Appendix B

Analysis of three-phase systems using linear transformations

In this chapter, the possibilities of using linear transformations in order to simplify the analysis of three-phase systems, are briefly discussed. These transformations are general and are valid under both symmetrical and un-symmetrical conditions. By generalizing the expressions for a symmetric three-phase voltage given in equations (2.10) and (2.14), corresponding expressions for an arbitrary three-phase voltage at constant frequency can be obtained as

$$\begin{array}{rcl} u_a(t) & = & U_{Ma}\cos(\omega t+\gamma_a) \\ u_b(t) & = & U_{Mb}\cos(\omega t+\gamma_b) \\ u_c(t) & = & U_{Mc}\cos(\omega t+\gamma_c) \end{array} \qquad \begin{array}{rcl} \overline{U}_a & = & U_a\angle\gamma_a^\circ \\ \overline{U}_b & = & U_b\angle\gamma_b^\circ \\ \overline{U}_c & = & U_c\angle\gamma_c^\circ \end{array} \tag{B.1}$$

where U_{Ma}, U_{Mb}, U_{Mc} are peak values, U_a, U_b, U_c are RMS-values and γ_a, γ_b, γ_c are phase angles of the three voltages. For the un-symmetrical currents, corresponding expressions hold as

$$\begin{array}{rcl} i_a(t) & = & I_{Ma}\cos(\omega t+\gamma_a-\phi_a) \\ i_b(t) & = & I_{Mb}\cos(\omega t+\gamma_b-\phi_b) \\ i_c(t) & = & I_{Mc}\cos(\omega t+\gamma_c-\phi_c) \end{array} \qquad \begin{array}{rcl} \overline{I}_a & = & I_a\angle\gamma_a^\circ-\phi_a \\ \overline{I}_b & = & I_b\angle\gamma_b^\circ-\phi_b \\ \overline{I}_c & = & I_c\angle\gamma_c^\circ-\phi_c \end{array} \tag{B.2}$$

where I_{Ma}, I_{Mb}, I_{Mc} are peak values, I_a, I_b, I_c are RMS-values of the three phase currents whereas ϕ_a, ϕ_b, ϕ_c are the phase of the currents in relation to the corresponding phase voltage.

The mean value of the total three-phase active power can be calculated as

$$P_3=\frac{U_{Ma}}{\sqrt{2}}\frac{I_{Ma}}{\sqrt{2}}\cos\phi_a+\frac{U_{Mb}}{\sqrt{2}}\frac{I_{Mb}}{\sqrt{2}}\cos\phi_b+\frac{U_{Mc}}{\sqrt{2}}\frac{I_{Mc}}{\sqrt{2}}\cos\phi_c \tag{B.3}$$

whereas the total three-phase complex power is

$$\begin{array}{rcl} \overline{S}_3 & = & \overline{U}_a\overline{I}_a^*+\overline{U}_b\overline{I}_b^*+\overline{U}_c\overline{I}_c^*=(U_aI_a\cos\phi_a+U_bI_b\cos\phi_b+U_cI_c\cos\phi_c)+ \\ & + & j(U_aI_a\sin\phi_a+U_bI_b\sin\phi_b+U_cI_c\sin\phi_c) \end{array} \tag{B.4}$$

This phase representation is in many cases sufficient for a three-phase system analysis. There are a number of important cases when the analysis can greatly be simplified by using linear transformations.

This chapter discusses the following items. First, the advantages of using linear transformations in three-phase system analysis are generally discussed. Later on, some specific transformations to be used in certain conditions are given. In order to really understand the subject of transformations, the reader is referred to text books on the subject, e.g. in electric machine theory or high power electronics. In chapter 8, one of the transformations of interest, symmetrical components, is discussed in more detail. The purpose of chapter **??** is to show that the idea and the mathematics behind the transformations are the same. It is only the choice of linear transformation, i.e. transformation matrix, that is different.

B.1 Linear transformations

By using transformations, components are mapped from an *original space* (the original space is here the instantaneous values or the complex representation of the phase quantities) to an *image space*. A *linear* transformation means that the components in the image space are a linear combination of the original space. The complex values of the phase voltages can be mapped with a linear transformation as

$$\begin{aligned} \overline{U}_A &= w_{aa}\overline{U}_a + w_{ab}\overline{U}_b + w_{ac}\overline{U}_c \\ \overline{U}_B &= w_{ba}\overline{U}_a + w_{bb}\overline{U}_b + w_{bc}\overline{U}_c \\ \overline{U}_C &= w_{ca}\overline{U}_a + w_{cb}\overline{U}_b + w_{cc}\overline{U}_c \end{aligned} \tag{B.5}$$

which in matrix form can be written as

$$\begin{bmatrix} \overline{U}_A \\ \overline{U}_B \\ \overline{U}_C \end{bmatrix} = \begin{bmatrix} w_{aa} & w_{ab} & w_{ac} \\ w_{ba} & w_{bb} & w_{bc} \\ w_{ca} & w_{cb} & w_{cc} \end{bmatrix} \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} \tag{B.6}$$

or in a more compact notation

$$\mathbf{U_{ABC}} = \mathbf{W}\mathbf{U_{abc}} \tag{B.7}$$

The elements in matrix $\mathbf{W}$ are independent of the values of the original and image space components. In this example, the components in the original space $\overline{U}_a$, $\overline{U}_b$ and $\overline{U}_c$ are mapped by using the linear transformation $\mathbf{W}$ to the image space components $\overline{U}_A$, $\overline{U}_B$ and $\overline{U}_C$. The original space components can be calculated from the image space components by using the inverse of matrix $\mathbf{W}$ ($\mathbf{W^{-1}} = \mathbf{T}$), i.e.

$$\mathbf{U_{abc}} = \mathbf{W^{-1}}\mathbf{U_{ABC}} = \mathbf{T}\mathbf{U_{ABC}} \tag{B.8}$$

The only mappings that are of interest, are those where $\mathbf{W^{-1}}$ are existing. In the following, the matrix $\mathbf{T}$ or its inverse $\mathbf{T^{-1}}$ will represent the linear transformation.

B.1.1 Power invarians

A usual demand for the linear transformations in power system analysis is that it should be possible to calculate the electric power in the image space by using the same expressions as in the original space and that the two spaces should give the same result. A transformation that can meet that requirement is called power invariant. Using the complex representation, the electric power in the original space can be calculated by using equation (B.4), this gives

$$\overline{S}_{abc} = \overline{U}_a\overline{I}_a^* + \overline{U}_b\overline{I}_b^* + \overline{U}_c\overline{I}_c^* = \mathbf{U^t_{abc}}\,\mathbf{I^*_{abc}} \tag{B.9}$$

where "$\mathbf{t}$" indicates the transpose.
In the image space, the corresponding expression is

$$\overline{S}_{ABC} = \overline{U}_A\overline{I}_A^* + \overline{U}_B\overline{I}_B^* + \overline{U}_C\overline{I}_C^* = \mathbf{U^t_{ABC}}\mathbf{I^*_{ABC}} \tag{B.10}$$

Power invarians implies that $\overline{S}_{ABC} = \overline{S}_{abc}$, i.e.

$$\mathbf{U^t_{ABC}I^*_{ABC}} = \mathbf{U^t_{abc}I^*_{abc}} = (\mathbf{TU_{ABC}})^{\mathbf{t}}(\mathbf{TI_{ABC}})^* = \mathbf{U^t_{ABC}T^tT^*I^*_{ABC}} \tag{B.11}$$

This gives that the transformation matrix $\mathbf{T}$ must fulfill the following condition :

$$\mathbf{T^tT^*} = ((\mathbf{T}^*)^{\mathbf{t}}\,\mathbf{T})^{\mathbf{t}} = \underline{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.12}$$

which leads to

$$\mathbf{T^{-1}} = (\mathbf{T}^*)^{\mathbf{t}} \tag{B.13}$$

If $\mathbf{T}$ is real, equation (B.13) implies that $\mathbf{T}$ is an orthogonal matrix.

B.1.2 The coefficient matrix in the original space

Consider a three-phase line between two buses. The voltage drop $\mathbf{U_{abc}}$ over the line depends on the current $\mathbf{I_{abc}}$ flowing in the different phases. The voltage drop can be expressed as

$$\mathbf{U_{abc}} = \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} = \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ac} \\ \overline{Z}_{ba} & \overline{Z}_{bb} & \overline{Z}_{bc} \\ \overline{Z}_{ca} & \overline{Z}_{cb} & \overline{Z}_{cc} \end{bmatrix} \begin{bmatrix} \overline{I}_a \\ \overline{I}_b \\ \overline{I}_c \end{bmatrix} = \mathbf{Z_{abc}I_{abc}} \tag{B.14}$$

where $\mathbf{Z_{abc}}$ is the *coefficient matrix* of the line. Note that each element in $\mathbf{Z_{abc}}$ is non-zero since a current in one phase has influence on the voltage drop in the other phases owing to the mutual inductance, see chapter 8.2.

Symmetrical matrices

A matrix that is symmetrical around its diagonal is called a symmetrical matrix (or more precisely, Hermitian if the matrix contains complex entries). For the Z-bus matrix in equation (B.14), this implies that $\overline{Z}_{ab} = \overline{Z}_{ba}$, $\overline{Z}_{ac} = \overline{Z}_{ca}$ and $\overline{Z}_{bc} = \overline{Z}_{cb}$, i.e.

$$\mathbf{Z_{abc}} = \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ac} \\ \overline{Z}_{ab} & \overline{Z}_{bb} & \overline{Z}_{bc} \\ \overline{Z}_{ac} & \overline{Z}_{bc} & \overline{Z}_{cc} \end{bmatrix} = \mathbf{Z^t_{abc}} \tag{B.15}$$

An example of a symmetrical matrix is the one representing a line (or a cable) where the non-diagonal element are dependent on the mutual inductance, which is equal between the phases a–b and the phases b–a, see chapter 8.2.

Cyclo-symmetrical matrices

The Z-bus matrix in equation (B.14) is cyclo-symmetric if $\overline{Z}_{ab} = \overline{Z}_{bc} = \overline{Z}_{ca}$, $\overline{Z}_{ba} = \overline{Z}_{ac} = \overline{Z}_{cb}$ and $\overline{Z}_{aa} = \overline{Z}_{bb} = \overline{Z}_{cc}$, i.e.

$$\mathbf{Z_{abc}} = \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ba} \\ \overline{Z}_{ba} & \overline{Z}_{aa} & \overline{Z}_{ab} \\ \overline{Z}_{ab} & \overline{Z}_{ba} & \overline{Z}_{aa} \end{bmatrix} \tag{B.16}$$

All normal three-phase systems are cyclo-symmetrical, i.e. if i_a, i_b, i_c are permuted to i_b, i_c, i_a, the voltages u_a, u_b, u_c will also be permuted to u_b, u_c, u_a. This implies that ordinary overhead lines, cables, transformers and electrical machines can be represented by cyclo-symmetrical matrices.

B.1.3 The coefficient matrix in the image space

If both sides of equation (B.14) are multiplied with the matrix $\mathbf{T^{-1}}$, the following is obtained

$$\mathbf{U_{ABC}} = \mathbf{T^{-1}U_{abc}} = \mathbf{T^{-1}Z_{abc}I_{abc}} = (\mathbf{T^{-1}Z_{abc}T})\mathbf{I_{ABC}} = \mathbf{Z_{ABC}I_{ABC}} \tag{B.17}$$

where

$$\mathbf{Z_{ABC}} = \mathbf{T^{-1}Z_{abc}T} \tag{B.18}$$

$\mathbf{Z_{ABC}}$ is the image space mapping of the coefficient matrix $\mathbf{Z_{abc}}$. This gives that if $\mathbf{U_{ABC}}$ represents the image space voltages, and $\mathbf{I_{ABC}}$ represents the image space currents then $\mathbf{Z_{ABC}}$ will represent the impedances in the image space.

One reason of introducing a linear transformation may be to obtain a diagonal coefficient matrix in the image space, i.e.

$$\mathbf{Z_{ABC}} = \begin{bmatrix} \overline{Z}_{AA} & 0 & 0 \\ 0 & \overline{Z}_{BB} & 0 \\ 0 & 0 & \overline{Z}_{CC} \end{bmatrix} \tag{B.19}$$

By having a diagonal coefficient matrix, equation (B.17) can be rewritten as

$$\begin{aligned} \overline{U}_A &= \overline{Z}_{AA}\overline{I}_A \\ \overline{U}_B &= \overline{Z}_{BB}\overline{I}_B \\ \overline{U}_C &= \overline{Z}_{CC}\overline{I}_C \end{aligned} \tag{B.20}$$

i.e. the matrix equation (B.17) with mutual couplings between the phases is replaced by three un-coupled equations. If $\mathbf{Z_{ABC}}$ is diagonal as in equation (B.19), both sides in equation (B.18) can be multiplied with $\mathbf{T}$ and rewritten as

$$\begin{aligned} \mathbf{TZ_{ABC}} &= \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix} \begin{bmatrix} \overline{Z}_{AA} & 0 & 0 \\ 0 & \overline{Z}_{BB} & 0 \\ 0 & 0 & \overline{Z}_{CC} \end{bmatrix} = \begin{bmatrix} \overline{Z}_{AA}T_1 & \overline{Z}_{BB}T_2 & \overline{Z}_{CC}T_3 \end{bmatrix} = \\ &= \mathbf{Z_{abc}T} = \mathbf{Z_{abc}} \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix} \end{aligned} \tag{B.21}$$

where T_1, T_2, T_3 are the columns of $\mathbf{T}$. Equation (B.21) can be rewritten as

$$\begin{aligned} \mathbf{Z_{abc}}T_1 - \overline{Z}_{AA}T_1 &= 0 \\ \mathbf{Z_{abc}}T_2 - \overline{Z}_{BB}T_2 &= 0 \\ \mathbf{Z_{abc}}T_3 - \overline{Z}_{CC}T_3 &= 0 \end{aligned} \tag{B.22}$$

i.e. $\overline{Z}_{AA}$, $\overline{Z}_{BB}$ and $\overline{Z}_{CC}$ are the eigenvalues of matrix $\mathbf{Z_{abc}}$ and the vectors T_1, T_2 and T_3 are the corresponding eigenvectors. A transformation that maps the matrix $\mathbf{Z}$ to a diagonal form should have a transformation matrix $\mathbf{T}$ having columns that are the eigenvectors of the matrix $\mathbf{Z}$. Note that eigenvectors can be scaled arbitrarily.

B.2 Examples of linear transformations that are used in analysis of three-phase systems

In the following, four commonly used linear transformations will be briefly introduced. In general, transformations can be presented in a little bit different way in different text books. It is therefore of importance to understand the definitions used by the authors.

B.2.1 Symmetrical components

In the analysis of un-symmetrical conditions in a power system, *symmetrical components* are commonly used. This complex, linear transformation uses the fact that all components (lines, machines, etc.) in normal systems are cyclo-symmetrical, i.e. their impedances can be modeled by equation (B.16). The *power invariant* transformation matrix and its inverse for the symmetrical components are

$$\mathbf{T_S} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \qquad \mathbf{T_S^{-1}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \tag{B.23}$$

where $\alpha = e^{j120^\circ}$. As given by the definition $\mathbf{T_S^{-1}} = (\mathbf{T_S^*})^\mathbf{t}$ which corresponds to the assumption of power invariant according to equation (B.13). By using this transformation, cyclo-symmetrical matrices are transformed into a diagonal form as given in equation (B.19), i.e. the columns of matrix $\mathbf{T_S}$ consist of the eigenvectors to a cyclo-symmetrical matrix. This will simplify the system analysis as indicated in equation (B.20). Using the given phasor voltages $\overline{U}_a$, $\overline{U}_b$ and $\overline{U}_c$, the *power invariant* symmetrical components can be calculated as

$$\mathbf{U_s} = \begin{bmatrix} \overline{U}_0 \\ \overline{U}_1 \\ \overline{U}_2 \end{bmatrix} = \mathbf{T_S^{-1}} \mathbf{U_{abc}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} \overline{U}_a \\ \overline{U}_b \\ \overline{U}_c \end{bmatrix} \tag{B.24}$$

The three components $\overline{U}_0$, $\overline{U}_1$ and $\overline{U}_2$ are called zero-sequence, positive-sequence and negative-sequence, respectively. A cyclo-symmetrical impedance matrix according to equation (B.16) can be diagonalized by using symmetrical components according to equation (B.18) as

$$\mathbf{Z_S} = \mathbf{T_S^{-1}} \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ba} \\ \overline{Z}_{ba} & \overline{Z}_{aa} & \overline{Z}_{ab} \\ \overline{Z}_{ab} & \overline{Z}_{ba} & \overline{Z}_{aa} \end{bmatrix} \mathbf{T_S} = \begin{bmatrix} \overline{Z}_0 & 0 & 0 \\ 0 & \overline{Z}_1 & 0 \\ 0 & 0 & \overline{Z}_2 \end{bmatrix} \tag{B.25}$$

where

$$\begin{array}{rcll} \overline{Z}_0 & = & \overline{Z}_{aa} + \overline{Z}_{ab} + \overline{Z}_{ba} = & \text{zero-sequence impedance} \\ \overline{Z}_1 & = & \overline{Z}_{aa} + \alpha^2 \overline{Z}_{ab} + \alpha \overline{Z}_{ba} = & \text{positive-sequence impedance} \\ \overline{Z}_2 & = & \overline{Z}_{aa} + \alpha \overline{Z}_{ab} + \alpha^2 \overline{Z}_{ba} = & \text{negative-sequence impedance} \end{array} \quad \text{(B.26)}$$

The three impedances $\overline{Z}_0$, $\overline{Z}_1$ and $\overline{Z}_2$ are the eigenvalues of the cyclo-symmetrical impedance matrix. For an impedance matrix that is both cyclo-symmetric and symmetric, i.e. $\overline{Z}_{ba} = \overline{Z}_{ab}$, the result after a diagonalization will be that

$$\begin{array}{rcl} \overline{Z}_0 & = & \overline{Z}_{aa} + 2\overline{Z}_{ab} \\ \overline{Z}_1 & = & \overline{Z}_{aa} - \overline{Z}_{ab} \\ \overline{Z}_2 & = & \overline{Z}_{aa} - \overline{Z}_{ab} \end{array} \quad \text{(B.27)}$$

Transformers, overhead lines, cables and symmetrical loads (not electrical machines) can be normally represented by impedance matrices that are both symmetrical and cyclo-symmetrical, i.e. all diagonal elements are equal and all non-diagonal elements are equal. This gives that the positive-sequence impedance and the negative-sequence impedance are equal.

In order to make the positive-sequence phasor voltage equal to the line-to-neutral phasor voltage, a *reference invariant* form of transformation for the symmetrical components is normally used. The *reference invariant* transformation matrix and its inverse are

$$\mathbf{T_{S'}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} = \sqrt{3} \cdot \mathbf{T_S} \qquad \mathbf{T_{S'}^{-1}} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot \mathbf{T_S^{-1}} \quad \text{(B.28)}$$

The reference invariant transformation is not power invariant since $\mathbf{T_{S'}^{-1}} = \frac{1}{3}(\mathbf{T_{S'}^{*}})^{\mathbf{t}}$. The name *reference invariant* means that in symmetrical conditions $\overline{U}_1 = \overline{U}_a$. Note that transformations of coefficient matrices, according to equation (B.18), are not influenced whether the power invariant or the reference invariant matrix is used since

$$\begin{array}{rcl} \mathbf{Z_{ABC}}(eff - inv) & = & \mathbf{T_S^{-1}} \mathbf{Z_{abc}} \mathbf{T_S} = \left(\frac{1}{\sqrt{3}} \mathbf{T_S^{-1}} \right) \mathbf{Z_{abc}} \left(\sqrt{3} \mathbf{T_S} \right) = \\ & = & \mathbf{T_{S'}^{-1}} \mathbf{Z_{abc}} \mathbf{T_{S'}} = \mathbf{Z_{ABC}}(ref - inv) \end{array} \quad \text{(B.29)}$$

A third variation of the transformation matrix for the symmetrical components arises when the ordering of the sequences is changed. If the positive-sequence is given first and the zero-sequence last, the columns of the $\mathbf{T}$-matrix and the rows in the $\mathbf{T^{-1}}$ are permuted, respectively. This results in the following *reference invariant* transformations matrices :

$$\mathbf{T_{S''}} = \begin{bmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha & 1 \\ \alpha & \alpha^2 & 1 \end{bmatrix} \qquad \mathbf{T_{S''}^{-1}} = \frac{1}{3} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \\ 1 & 1 & 1 \end{bmatrix} \quad \text{(B.30)}$$

This form of the transformation matrices will be used in chapter 8 where symmetrical components are discussed in more detail. The only thing that happens with the coefficient matrix in the image space is that the diagonal elements change places.

As described above, a number of different variations of the symmetrical components can be used, all having the same fundamental purpose, to diagonalize the cyclo-symmetrical impedance matrices.

B.2.2 Clarke's components

Clarke's components, also called $\alpha-\beta$-components or orthogonal components, divides those phase-quantities not having any zero-sequence into two orthogonal components. The word *zero-sequence* means the same as when discussing symmetrical components, the sum of the phase components. Components that do not have zero-sequence are those whose sum is equal to zero. The *power invariant* ($\mathbf{T^{-1}} = (\mathbf{T}^*)^\mathbf{t}$, see equation (B.13)) transformation matrix and its inverse for the Clarke's components are

$$\mathbf{T_C} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \qquad \mathbf{T_C^{-1}} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \tag{B.31}$$

Clarke's components are a real orthogonal transformation that is mainly used in transformations of time quantities, e.g.

$$\begin{bmatrix} i_0(t) \\ i_\alpha(t) \\ i_\beta(t) \end{bmatrix} = \mathbf{i_{0\alpha\beta}(t)} = \mathbf{T_C^{-1} i_{abc}(t)} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} \tag{B.32}$$

where $i_0(t)$ is the zero-sequence component, $i_\alpha(t)$ is the α-component and i_β is the β-component for Clarke's transformation of the phase currents $i_a(t)$, $i_b(t)$ and $i_c(t)$. The reason why the transformation is orthogonal is given by column two and three of $\mathbf{T_C}$ (corresponds to the α- and β-components) since the columns are orthogonal.

For a symmetrical three-phase current given by equation (2.12)

$$\begin{aligned} i_a(t) &= I_M \cos(\omega t - \phi) \\ i_b(t) &= I_M \cos(\omega t - 120^\circ - \phi) \\ i_c(t) &= I_M \cos(\omega t + 120^\circ - \phi) \end{aligned} \tag{B.33}$$

the Clarke's components are given by equation (B.32)

$$\begin{aligned} i_0(t) &= \frac{1}{\sqrt{3}} \left(i_a(t) + i_b(t) + i_c(t) \right) = 0 \\ i_\alpha(t) &= \sqrt{\frac{2}{3}} \left(i_a(t) - \frac{1}{2} i_b(t) - \frac{1}{2} i_c(t) \right) = \sqrt{\frac{3}{2}} I_M \cos(\omega t - \phi) \\ i_\beta(t) &= \sqrt{\frac{2}{3}} \left(\frac{\sqrt{3}}{2} i_b(t) - \frac{\sqrt{3}}{2} i_c(t) \right) = \sqrt{\frac{3}{2}} I_M \cos(\omega t - \phi - 90^\circ) \end{aligned} \tag{B.34}$$

As given above, conditions not having any zero-sequence can be fully represented by Clarke's α- and β-components. Conditions not having any zero-sequence are quite common and depends, among other things, on the type of transformer connection used.

Matrices that are both symmetrical and cyclo-symmetrical can be diagonalized by using

Clarke's transform as

$$\begin{aligned}
\mathbf{Z_C} &= \mathbf{T_C^{-1}} \begin{bmatrix} \overline{Z}_{aa} & \overline{Z}_{ab} & \overline{Z}_{ab} \\ \overline{Z}_{ab} & \overline{Z}_{aa} & \overline{Z}_{ab} \\ \overline{Z}_{ab} & \overline{Z}_{ab} & \overline{Z}_{aa} \end{bmatrix} \mathbf{T_C} = \\
&= \begin{bmatrix} \overline{Z}_{aa} + 2\overline{Z}_{ab} & 0 & 0 \\ 0 & \overline{Z}_{aa} - \overline{Z}_{ab} & 0 \\ 0 & 0 & \overline{Z}_{aa} - \overline{Z}_{ab} \end{bmatrix}
\end{aligned} \tag{B.35}$$

For this type of matrices, the diagonalization based on Clarke's components gives exactly the same answer as the diagonalization based on symmetrical components, see equations (B.25) and (B.27).

This gives that the matrix representation of transformers, overhead lines, cables and symmetrical loads (not electrical machines) can be diagonalized. The advantage of using Clarke's components is that the transformation is real which implies that the mapping of real instantaneous quantities are also real. The disadvantage is that electrical machines cannot be represented by three independent variables by using Clarke's components.

Clarke's components are used in order to simplify the analysis of e.g. multi-phase short circuits, transient system behavior, converter operation, etc.

B.2.3 Park's transformation

Park's transformation (also called dq-transformation or Blondell's transformation) is a linear transformation between the three physical phases and three new components. This transformation is often used when analyzing synchronous machines.

In Figure B.1, a simplified description of the internal conditions of a synchronous machine having salient poles, is given. Two orthogonal axis are defined. One is directed along the magnetic flux induced in the rotor. The second axis is orthogonal to the first axis. The first axis is called the direct-axis (d-axis) and the second axis is called the quadrature-axis (q-axis). Note that this system of coordinates follows the rotation of the rotor. The machine given in Figure B.1 is a two-pole machine, but Park's transformation can be used for machines having an arbitrary number of poles.

As indicated above, Park's transformation is time independent since the displacement between the dq-axes and the abc-axes is changed when the rotor revolves. The Park's transformation includes not only the d- and q-components, but also the zero-sequence in order to achieve a complete representation. The connection between phase currents i_a, i_b, and i_c and the dq0-components is given by the notation given in Figure B.1

$$\begin{aligned}
i_0 &= \frac{1}{\sqrt{3}} (i_a + i_b + i_c) \\
i_d &= \sqrt{\frac{2}{3}} (i_a \cos\beta + i_b \cos(\beta - 120°) + i_c \cos(\beta + 120°)) \\
i_q &= \sqrt{\frac{2}{3}} (i_a \sin\beta + i_b \sin(\beta - 120°) + i_c \sin(\beta + 120°))
\end{aligned} \tag{B.36}$$

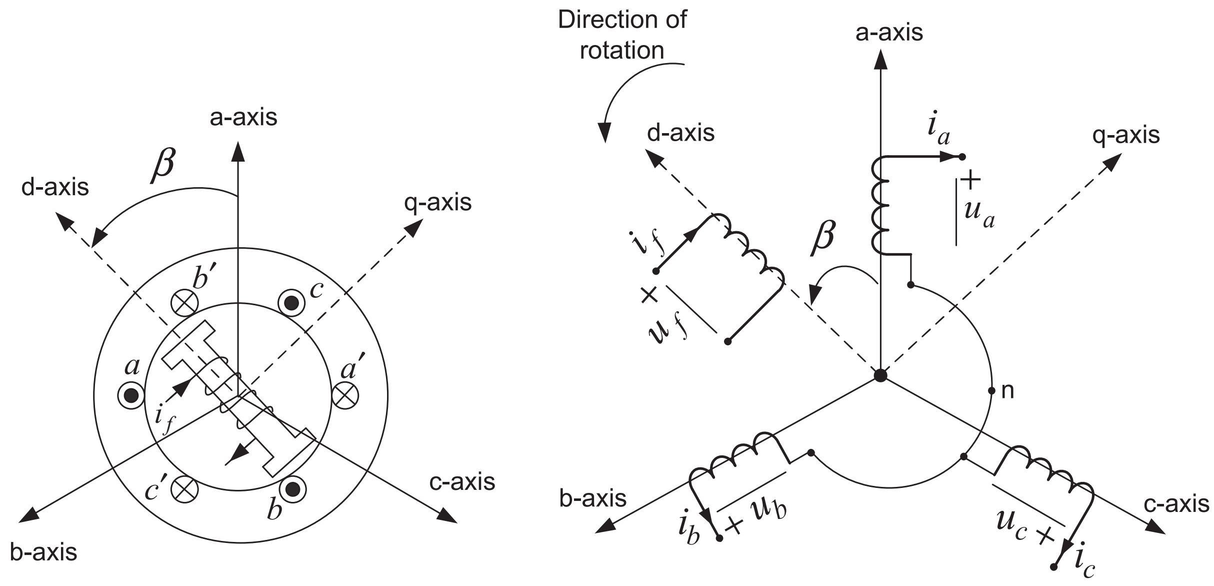


Figure B.1. Definitions of quantities in Park's transformation.

This equation can be written on matrix form as

$$\mathbf{i_{0dq}} = \begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\beta & \cos(\beta - 120°) & \cos(\beta + 120°) \\ \sin\beta & \sin(\beta - 120°) & \sin(\beta + 120°) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \mathbf{T_P^{-1} i_{abc}} \quad \text{(B.37)}$$

The matrix $\mathbf{T_P}$ and matrix $\mathbf{T_P^{-1}}$ can be transposed as

$$\mathbf{T_P} = (\mathbf{T_P^{-1}})^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\beta & \sin\beta \\ \frac{1}{\sqrt{2}} & \cos(\beta - 120°) & \sin(\beta - 120°) \\ \frac{1}{\sqrt{2}} & \cos(\beta + 120°) & \sin(\beta + 120°) \end{bmatrix} = (\mathbf{T_P^{-1}})^{\mathbf{t}} \quad \text{(B.38)}$$

The transformation is hence power invariant according to equation (B.13). Park's transformation is real and usable when transforming time quantities. Note that the Park's transformation is linear but the transformation matrix is time dependent. At constant frequency $\beta = \omega t + \beta_0$.

The Park's transformation is a frequency transformed version of Clarke's transformation. When $\beta = 0$ and the q-axis leads the d-axis, the transformation matrices are identical, i.e. $\mathbf{T_C} = \mathbf{T_P}(\beta = \mathbf{0})$.

B.2.4 Phasor components

Phasor components are mainly used at instantaneous value analysis when a single machine or when several machines are connected together. The *power invariant* ($\mathbf{T^{-1}} = (\mathbf{T}^*)^{\mathbf{t}}$, see equation (B.13)) transformation matrix and its inverse for these components are

$$\mathbf{T_R} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & e^{j\theta} & e^{-j\theta} \\ 1 & \alpha^2 e^{j\theta} & \alpha e^{-j\theta} \\ 1 & \alpha e^{j\theta} & \alpha^2 e^{-j\theta} \end{bmatrix} \tag{B.39}$$

$$\mathbf{T_R^{-1}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ e^{-j\theta} & \alpha e^{-j\theta} & \alpha^2 e^{-j\theta} \\ e^{j\theta} & \alpha^2 e^{j\theta} & \alpha e^{j\theta} \end{bmatrix}$$

where $\alpha = e^{j120^\circ}$. The phasor components of the three-phase currents $i_a(t)$, $i_b(t)$ and $i_c(t)$ can be obtained as

$$\begin{bmatrix} i_0(t) \\ \bar{i}_s(t) \\ \bar{i}_z(t) \end{bmatrix} = \mathbf{i_{0sz}(t)} = \mathbf{T_R^{-1} i_{abc}(t)} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ e^{-j\theta} & \alpha e^{-j\theta} & \alpha^2 e^{-j\theta} \\ e^{j\theta} & \alpha^2 e^{j\theta} & \alpha e^{j\theta} \end{bmatrix}\begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} \tag{B.40}$$

where $i_0(t)$ is the zero-sequence component and $\bar{i}_s(t)$ is called the field vector current, the complex phasor of the current. The current $\bar{i}_s(t)$ is complex since the transformation matrix is complex. By assuming that $i_a(t)$, $i_b(t)$ and $i_c(t)$ are real, the expression for $\bar{i}_z(t)$ can be written as

$$\begin{aligned} \bar{i}_z(t) &= \frac{1}{\sqrt{3}} e^{j\theta}\left(i_a(t) + \alpha^2 i_b(t) + \alpha i_c(t)\right) = \\ &= \left[\frac{1}{\sqrt{3}} e^{-j\theta}\left(i_a(t) + \alpha i_b(t) + \alpha^2 i_c(t)\right)\right]^* = \bar{i}_s^*(t) \end{aligned} \tag{B.41}$$

i.e. $\bar{i}_z(t)$ is known if the field vector $\bar{i}_s(t)$ is known. Under conditions of no zero-sequence components, the field vector is fully describing an arbitrary real three-phase quantity.

For a symmetrical three-phase current as given in equation (B.33), the phasor components can be obtained according to equation (B.40)

$$\begin{aligned} i_0(t) &= \frac{1}{\sqrt{3}}\left(i_a(t) + i_b(t) + i_c(t)\right) = 0 \\ \bar{i}_s(t) &= \frac{e^{-j\theta}}{\sqrt{3}}\left(i_a(t) + \alpha i_b(t) + \alpha^2 i_c(t)\right) = \frac{\sqrt{3}}{2} I_M e^{j(\omega t - \phi - \theta)} \\ \bar{i}_z(t) &= \frac{e^{j\theta}}{\sqrt{3}}\left(i_a(t) + \alpha^2 i_b(t) + \alpha i_c(t)\right) = \frac{\sqrt{3}}{2} I_M e^{-j(\omega t - \phi - \theta)} = \bar{i}_s^*(t) \end{aligned} \tag{B.42}$$

Finally, for $\theta = \omega t$ the following is obtained

$$\begin{aligned} i_0(t) &= 0 \\ \bar{i}_s(t) &= \frac{\sqrt{3}}{2} I_M e^{-j\phi} \\ \bar{i}_z(t) &= \frac{\sqrt{3}}{2} I_M e^{j\phi} = \bar{i}_s^*(t) \end{aligned} \tag{B.43}$$

i.e. the field vector current $i_s(t)$ has a constant magnitude, independent of time.

By assuming real phase currents having no zero-sequence, they can be calculated by using the field vector current as

$$\begin{aligned}\begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} &= \mathbf{T_R i_{0sz}(t)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & e^{j\theta} & e^{-j\theta} \\ 1 & \alpha^2 e^{j\theta} & \alpha e^{-j\theta} \\ 1 & \alpha e^{j\theta} & \alpha^2 e^{-j\theta} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{i}_s(t) \\ \bar{i}_s^*(t) \end{bmatrix} = \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} \left[e^{j\theta}\bar{i}_s(t)\right] + \left[e^{j\theta}\bar{i}_s(t)\right]^* \\ \left[\alpha^2 e^{j\theta}\bar{i}_s(t)\right] + \left[\alpha^2 e^{j\theta}\bar{i}_s(t)\right]^* \\ \left[\alpha e^{j\theta}\bar{i}_s(t)\right] + \left[\alpha^2 e^{j\theta}\bar{i}_s(t)\right]^* \end{bmatrix} = \frac{2}{\sqrt{3}} \begin{bmatrix} Re\left[e^{j\theta}\bar{i}_s(t)\right] \\ Re\left[\alpha^2 e^{j\theta}\bar{i}_s(t)\right] \\ Re\left[\alpha e^{j\theta}\bar{i}_s(t)\right] \end{bmatrix}\end{aligned} \tag{B.44}$$

Phasor components are a frequency transformed form of the symmetrical components. For $\theta = 0$ (phasor components), the transformation matrix for the phasor components and the symmetrical components are identical, i.e. $\mathbf{T_R}(\theta = \mathbf{0}) = \mathbf{T_S}$.