lecture1: A short introduction to Arnold diffusion and splitting of separatrices

Master Class

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Stable and unstable manifolds of saddle points

Some know facts:
Given $\dot{x} = X(x)$, X is smooth enough, $x \in \mathbb{R}^n$. Denote $\varphi(t; x)$ the flow (solution such that $\varphi(0; x) = x$)

- Consider a critical point x^* $(X(x^*) = 0,\ \varphi(t; x^*) = x^*)$ such that $DX(x^*)$ has eigenvalues with positive real part and eigenvalues with negative real part it is called a saddle (hyperbolic point).
- The stable and unstable manifold Theorem says that exist two manifolds $W^u_{loc}(x^*)$, $W^s_{loc}(x^*)$ defined in a neigborhood $U \in \mathbb{R}^n$ of x^* such that
  1. They are invariant by the flow: if $x \in W^{u,s}_{loc}(x^*)$ then the orbit $\varphi(t; x) \in W^{u,s}_{loc}(x^*)$ for all $t \in \mathbb{R}$ such that $\varphi(t; x) \in U$.
  2. $W^{u,s}_{loc}(x^*)$ contain the initial conditions of points whose orbit is asymptotic to x^*.
  3. If $x \in W^s_{loc}(x^*)$ then $\varphi(t; x) \to x^*$ when $t \to +\infty$
  4. If $x \in W^u_{loc}(x^*)$ then $\varphi(t; x) \to x^*$ when $t \to -\infty$
  5. They are as smooth as the flow and depend regularly on parameters.

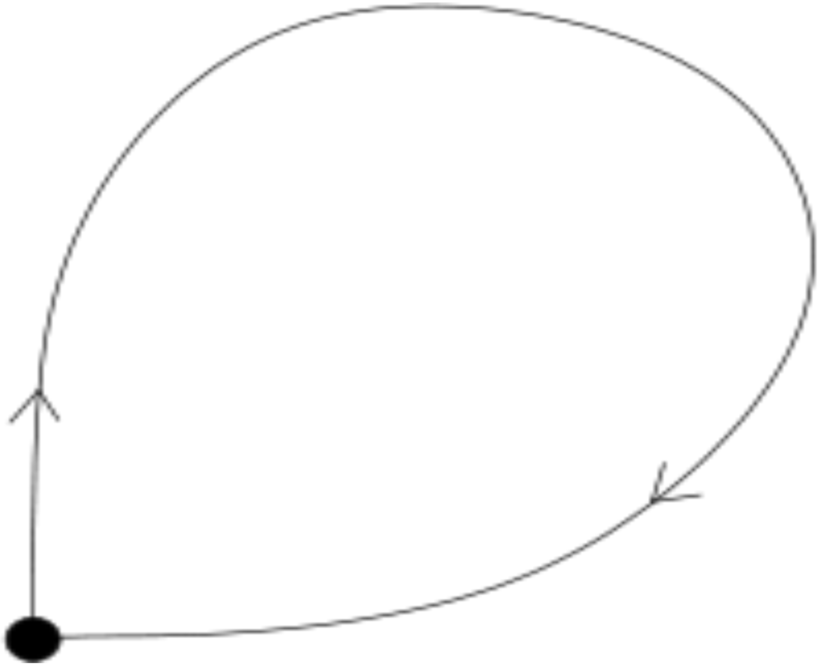
Stable and unstable manifolds of saddle points

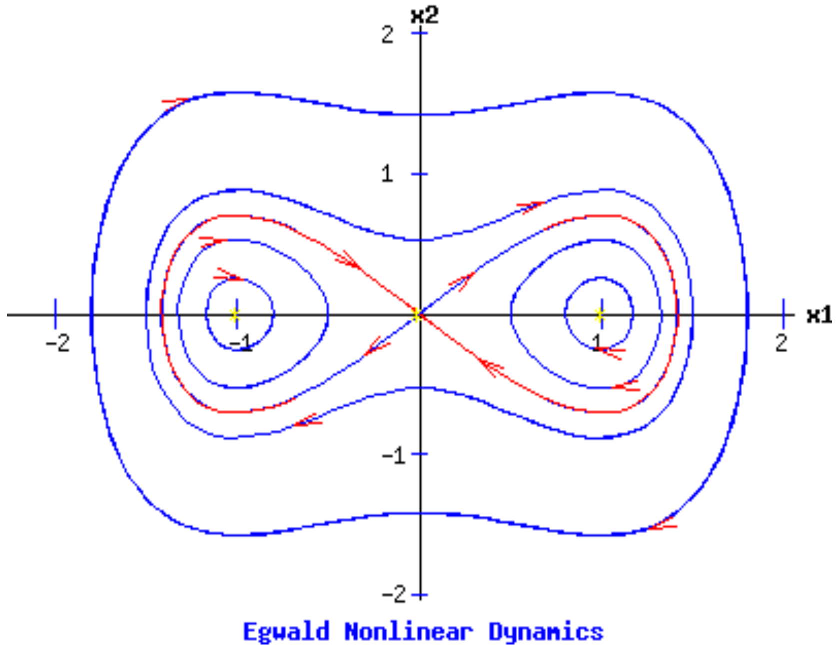
- The stable and unstable manifolds can be globalized:
  1. $W^u(x^*) = \cup_{t\geq 0}\{\varphi(t,x),\ x \in W^u_{loc}(x^*)\}$
  2. $W^s(x^*) = \cup_{t\leq 0}\{\varphi(t,x),\ x \in W^s_{loc}(x^*)\}$
- The global stable and unstable manifolds can intersect. We call a point x_h homoclinic if $x_h \in W^u(x^*) \cap W^s(x^*)$. (heteroclinic if $x_h \in W^u(x_1^*) \cap W^s(x_2^*)$, and x_i^*, $i = 1, 2$ are fixed points).
- If x_h is an homoclinic point, then the whole orbit of x_h is in the intersection: $\varphi(t; x_h) \subset W^u(x^*) \cap W^s(x^*)$, $\forall t \in \mathbb{R}$
- In flows, there is an analogous phenomenon for periodic orbits, invariant tori, etc
- There is an analogous phenomenon for diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ and hyperbolic fixed points with $Df(x^*)$ has eigenvalues with modulus bigger or smaller than 1.
- If x_h is an homoclinic point, then the whole orbit of x_h is in the intersection: $\{f^n(x_h)\} \subset W^u(x^*) \cap W^s(x^*)$, $\forall n \in \mathbb{Z}$.

Splitting of separatrices.

- One interesting problem is to show that the stable and unstable manifolds intersect transversally.
- There are methods to compute good approximations of the local invariant manifolds of the hyperbolic point (or periodic orbit, etc) $W^{u,s}_{loc}$.
- It is almost impossible to compute the global ones
- There are some systems that are integrable (for instance Hamiltonian systems of one degree of freedom) and for these ones one can have explicit formulas for the manifolds.
- In integrable Hamiltonian systems, the manifolds are coincident giving rise to the so called homoclinic orbits.

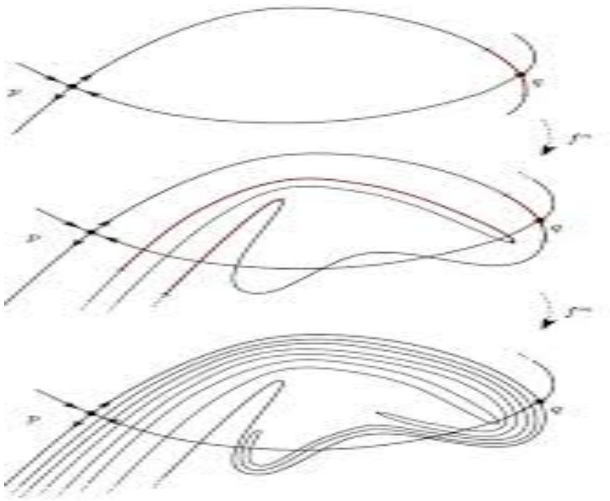
Homoclinic orbits.

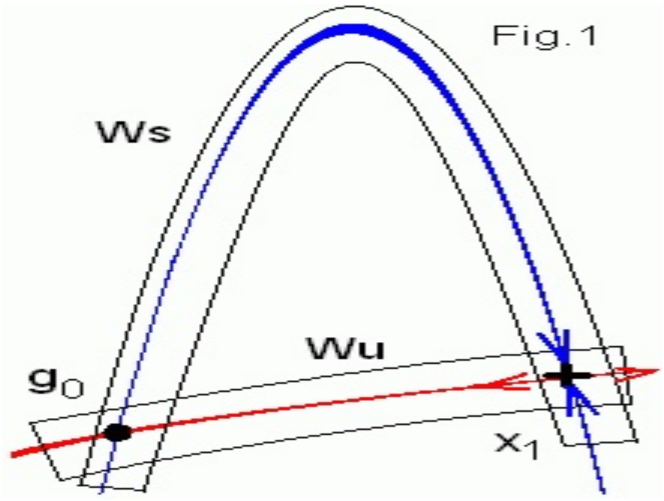




Splitting of separatrices and Chaos

- It is interesting to find trasversal intersections of estable and unstable manifolds, for instance to create chaos.
- If there is a transversal intersection between the stable and unstable manifolds of a hyperbolic fixed point of a diffeomorphism of the plane one can see the existence of a horseshoe for a suitable iteration of the map.
- The horseshoe map provides the existence of symbolic dynamics.





Some basic facts in Hamiltonian Systems

A Hamiltonian system of n degrees of freedom is a system of differential equations of the form:

$$\left.\begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p) \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q,p) \end{array}\right\} \quad \dot{x} = J\nabla H(x);\; x = (q,p),\; J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix} \tag{1}$$

Where $H : U \to \mathbb{R}$ smooth, $U \subset \mathbb{R}^{2n}$ or $U \subset \mathbb{T}^n \times \mathbb{R}^n$.

- Sometimes H also depends periodically on time: $H = H(q,p,t)$, and then $(q,p,t) \in U \times \mathbb{T}$.
- In the autonomous case (H does not depent on time) H is a first integral of the system (constant of motion): $H(q(t),p(t)) = H(q(0),p(0))$ for any solution $(q(t),p(t)))$. Notation: $x = (q,p)$.

Non autonomous case: the Poincaré map

- In the non-autonomous case we can also work with the Poincaré map. We fix an initial time $t_0 = \theta$ and we have a map in U:

$$\mathcal{P}_\theta : U \subset \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$$

  given by $\mathcal{P}_\theta(x) = \Phi(\theta + 2\pi; \theta, x)$, where $\Phi(t; \theta, x)$ is the solution of the non autonomous Hamiltonian system such that $\Phi(\theta; \theta, x) = x$.
- In the autonomous case, as $\Phi(t; \theta, q, p; \varepsilon) = \varphi(t - \theta, x)$, where $\varphi(t, x)$ is the flow of the system, we have $\mathcal{P}_\theta(x) = \varphi(2\pi, x)$ is independent of θ.
- **Exercice**
  We can follow the orbis using $\mathcal{P}_\theta$:

$$\mathcal{P}_\theta^k(x) = \Phi(\theta + 2\pi k, \theta, x)$$

Goal: understand the global behaviour of the orbits.

Symplectic maps

- A symplectic map is a diffeomorphism that preserves a symplectic structure.
- The simplest example of symplectic map is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the area and orientation, i.e. such that for all points $x \in \mathbb{R}^2$ we have

$$\det DF(x) = 1$$

- In terms of differential forms, this can be expressed as

$$F^*(dq \wedge dp) = dq \wedge dp, \quad x = (p, q)$$

Symplectic maps

- More generally, if M and N are manifolds of dimension $2n$ and ω_M and ω_N are symplectic forms (non-degenerate, closed, differentiable 2-forms) on M and N, then a diffeomorphism $F : M \to N$ is a symplectic map if $F^*\omega_N = \omega_M$.
- By a theorem of Darboux, for each point $x \in M$ one can always find local coordinates such that this is equivalent to:

$$DF(x)^t JDF(x) = J,$$

where $J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & o \end{pmatrix}$

- The Poincaré map of a time dependent Hamiltonian system is a symplectic map.

Integrable Hamiltonian systems

Observe that if $H = H(p)$, the equations of motion are:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(p) \\ \dot{p} &= 0 \end{aligned} \tag{2}$$

Therefore the system can be integrated:

$$p(t) = p_0, \quad q(t) = q_0 + \frac{\partial H}{\partial p}(p_0)t,$$

If $q \in \mathbb{T}^n$ the motion is confined in n-dimensional tori:

$$\mathcal{T}_{p_0} = \{(q,p),\ p = p_0,\ q \in \mathbb{T}^n\}$$

As $\varphi(t, p_0, q_0) = (p_0, q_0 + \omega(p_0)t)$, where $\omega(p_0) = \frac{\partial H}{\partial p}(p_0) = \nabla H_0(p_0)$, the motion in the torus $\mathcal{T}_{p_0}$ is quasiperiodic with frequency $\omega(p_0)$.

Known facts

• A canonical transformation is a change of canonical coordinates (q, p) → (Q, P) that preserves the form of Hamilton's equations (that is, the new Hamilton's equations resulting from the transformed Hamiltonian may be simply obtained by substituting the new coordinates for the old coordinates). Any symplectic map gives a canonical transformation.
• A first integral of $\dot{x} = J\nabla H(x)$ is a smooth function

$$F : \mathcal{U} \to \mathbb{R}$$

where $\mathcal{U} \subset \mathbb{T}^n \times \mathbb{R}^n$ or $\mathcal{U} \subset \mathbb{R}^{2n}$ such that is constant on trajectories, that is, if $x(t)$ is a solution:

$$F(x(t)) = F(x(0))$$

• The Poisson bracket of two functions $F(q,p)$, $G(q,p)$ is given by:

$$\{F, G\}(q,p) = (\sum_{j=1}^{n} \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j})(q,p) = \nabla F^t J \nabla G x) = DFJ\nabla G(x)$$

Integrable Hamiltonian systems

THEOREM (Liouville-Arnold)
Let $H(x)$ be a Hamiltonian system with n degrees of freedom ($x = (q, p) \in \mathbb{R}^{2n}$), and assume there are also known n first integrals of motion $F_1 \dots F_n$ that are independent ($\nabla F_1, \dots \nabla F_n$ are independent as vectors) and are in involution:

$$\{F_i, F_j\} = 0$$

Then there exists a canonical transformation

$$(p, q) \to (I, \phi)$$

to action-angle coordinates in which the transformed Hamiltonian is dependent only upon the action coordinates.

$$H(q, p) = \mathcal{H}(I)$$

Near integrable systems

- We will work with non-autonomous systems close to integrable:
- $H(I, \phi, t; \varepsilon) = H_0(I) + \varepsilon H_1(I, \phi, t; \varepsilon)$, $(I, \phi, t) \in \mathbb{R}^n \times \mathbb{T}^{n+1}$,
- The corresponding Poincaré map in $\Sigma_\theta = \{(I, \phi, \theta)\}$ with $(I, \phi) \in \mathbb{R}^n \times \mathbb{T}^n$ given by $\mathcal{P}_\theta(I, \phi; \varepsilon) = \Phi(\theta + 2\pi; \theta, I, \phi; \varepsilon)$.
- When $\varepsilon = 0$, the motion is given by

$$\Phi(t; \theta, I_0, \phi_0; 0) = \varphi(t - \theta, I_0, \phi_0) = (I_0, \phi_0 + \nabla H_0(I_0)(t - \theta)),$$

- The Poincaré map

$$\mathcal{P}_\theta(I_0, \phi_0; 0) = (I_0, \phi_0 + \nabla H_0(I_0) 2\pi).$$

- The motion is confined in n-dimensional invariant tori:

$$\mathcal{T}_{I_0} = \{(I_0, \phi),\ \phi \in \mathbb{T}^n, \},$$

- As $\mathcal{P}_\theta(I_0, \phi_0; 0) = (I_0, \phi_0 + \nabla H_0(I_0) 2\pi)$, the dynamics for the Poincaré map is quasi-periodic with frequency $\omega(I_0) = (\nabla H_0(I_0))$.
- What happens when $\varepsilon \neq 0$?

Near integrable systems: Stability

- The stability result is given by KAM theorem.
- KAM theory shows that, under suitable regularity and non-degeneracy assumptions, most (in measure theoretic sense) of the tori $\mathcal{T}_{I_0}$ persist (slightly deformed) under small Hamiltonian perturbations.
- The union of persistent n-dimensional tori (Kolmogorov set) tend to fill the whole phase space as the strength of the perturbation is decreased.
- We will consider tori $\mathcal{T}_{I_0}$, $I_0 \in \mathbb{R}^n$ such that the frequency vector $\omega = \omega(I_0)$ is rationally independent and "badly" approximated by rationals, typically in a Diophantine sense:

$$\exists \gamma, \ \tau, \quad ||\omega \cdot \ k|| = |\sum_{j=1}^{n} \omega_i k_i| \geq \frac{\gamma}{||k||^{\tau}}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}$$

- These tori cover the phase space except a set of measure $\mathcal{O}(\sqrt{\gamma})$.

Near integrable systems: Stability

- Consider an integrable Hamiltonian system: $H_0(I)$ and assume that $\frac{\partial^2 H_0}{\partial^2 I}$ is invertible.
- Consider $I_0 \in \mathbb{R}^n$ such that the frequency vector $\omega = \omega(I_0) = \nabla H_0(I_0)$ is rationally independent and "badly" approximated by rationals, in a Diophantine sense with $\gamma = \varepsilon$.
- KAM theorem says that a Hamiltonian system of the form:

$$H(I, \phi; \varepsilon) = H_0(I) + \varepsilon H_1(I; \phi; \varepsilon),$$

has an invariant torus ε-close to $\mathcal{T}_{I_0}$.

- These tori cover the phase space except a set of measure $\mathcal{O}(\sqrt{\varepsilon})$.
- Resonances: values of I such that $\omega(I) \cdot \ k = 0$ for some $k \in \mathbb{Z}$
- The set not covered ny KAM tori is the union of balls centered at the resonances and of size $\mathcal{O}(\sqrt{\varepsilon})$.

Near integrable systems: Instability

$$H(I, \phi; \varepsilon) = H_0(I) + \varepsilon H_1(I; \phi; \varepsilon),$$

Now that we know that there are a lot of bounded (stable) motions close to the unperturbed ones, we want to understant the behavior of the rest of orbits of the system.

- Are they stable? are there some unstable motions?
- Lots of researchers are trying to answer this question since Arnold introduced his example in 1964 (we will show it later).
- We will see that the number of degrees of freedom plays a crucial role in the answer to this question.
- We will give tools to answer this question based in the so called geometric methods.

The case of one degree of freedom

- If $n = 1$, the perturbed Hamiltonian system is also integrable!.
- But not in the classical sense of Liouville-Arnold: we don't have global action-angle variables.
- If we consider $H(I, \phi; \varepsilon) = H_0(I) + \varepsilon H_1(I, \phi)$, with $(I, \phi) \in \mathbb{R} \times \mathbb{T}$. H is a first integral. In the level curves $H(I, \phi) = h$ we see the structure of the KAM theorem.

Example: the pendulum equation:

$$H(I, \phi; \varepsilon) = H_0(I) + \varepsilon H_1(I, \phi) = \frac{I^2}{2} + \varepsilon V(\phi) = \frac{I^2}{2} + \varepsilon(\cos(\phi) - 1)$$

Equations:

$$\begin{aligned} \dot{\phi} &= I \\ \dot{I} &= -\varepsilon V'(\phi) = \varepsilon \sin\phi \end{aligned}$$

The case of one degree of freedom

- For $\varepsilon = 0$, $H(I, \phi; 0) = H_0(I) = \frac{I^2}{2}$, the system is integrable.
- The 2-dimensional space is foliated by 1 dimensional tori (curves) of the flow:

$$\mathcal{T}_{I_0} = \{(I_0, \phi),\ \phi \in \mathbb{T}\}$$

- The flow in $\mathcal{T}_{I_0}$ is a rotation with frequency $\omega(I_0) = I_0$.

$$\varphi(t; I_0, \phi; 0) = (I_0, \phi + \omega(I_0)t).$$

- The same happens for the Poincaré map

$$\mathcal{P}_\theta(I_0, \phi) = (I_0, \phi + \omega(I_0)2\pi)$$

- Frequency: $\omega(I) = I$, resonance $I = 0$.

The case of one degree of freedom

For $\varepsilon > 0$, the level curves $H(I, \phi; \varepsilon) = \frac{I^2}{2} + \varepsilon(\cos(\phi) - 1) = h$ give the phase portrait of the system.

- When $h > 0$ is far from zero the level curves of H are close to the level curves of H_0: $I = I_0 + O(\varepsilon)$, $I_0 = \sqrt{2h}$
- $(0, 0)$ is a saddle and its stable and unstable manifolds coincide along a separatrix (homoclinic orbit)

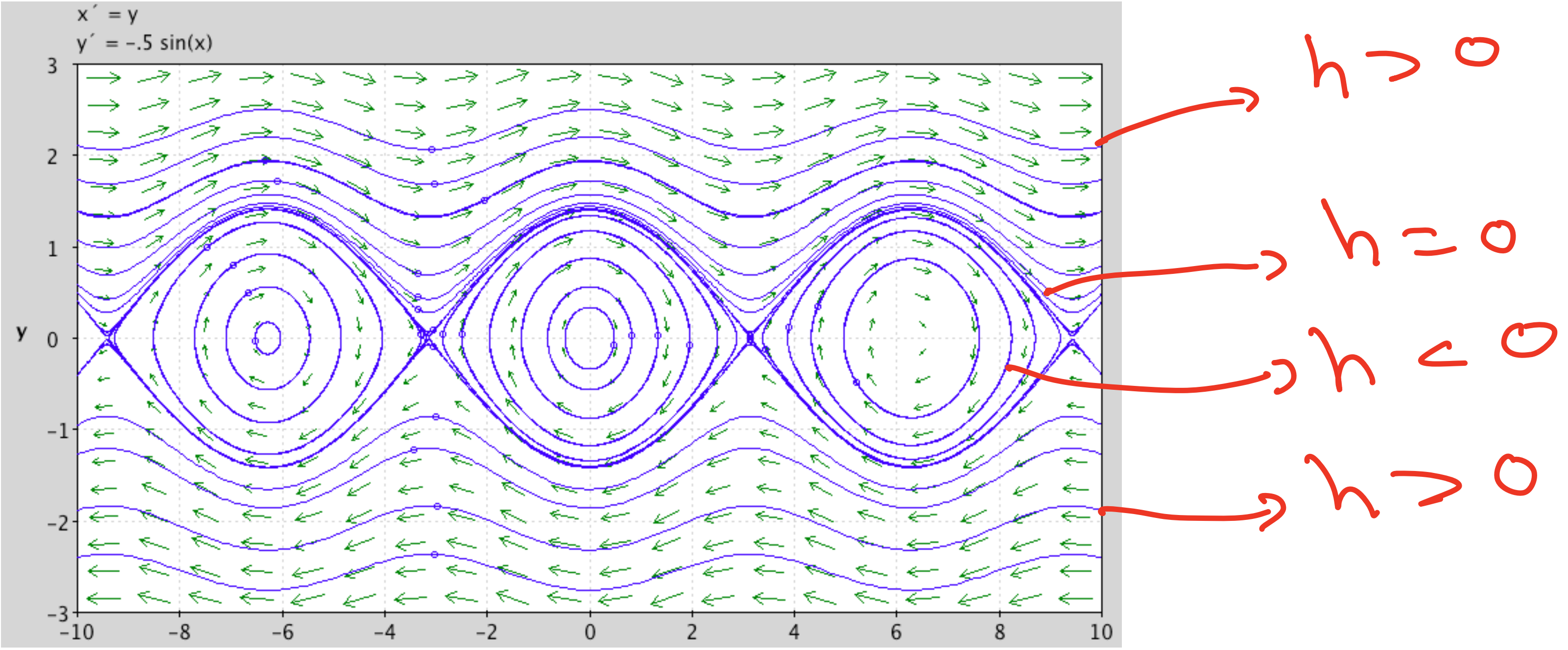
$$H(I, \phi; \varepsilon) = \frac{I^2}{2} + \varepsilon(\cos(\phi) - 1) = 0$$

  The tori close to $I = 0$ $(h = 0)$ have disapeared!

  The energy level $h = 0$ contains an equilibrium point of saddle type and its separatrices (stable and unstable manifolds which coincide due to the integrability of H).

  When $h < 0$ inside the separatrices of the saddle we have tori of different topology (contractible to a point).
- The Poincaré map has the same phase portrait as the flow: $\mathcal{P}_\theta(I_0, \phi_0) = \varphi(2\pi, I_0, \phi_0)$

The case of one degree of freedom



- Near the resonant value $I = 0$ the tori have been destroyed and "new" objects appear: tori of lower dimension with stable and unstable manifolds (whiskered tori) and tori of different toplology.
- Answer to the Instability question: No Instability, all the perturbed motions are close to the unperturbed ones. The action changes at most $\mathcal{O}(\sqrt{\varepsilon})$.

The case of one and a half degrees of freedom

If we consider $H(I,\phi,t;\varepsilon) = H_0(I) + \varepsilon H_1(I,\phi,t;\varepsilon)$, with $(I,\phi,t) \in \mathbb{R}\times\mathbb{T}^2$, equations:

$$\begin{aligned} \dot\phi &= \frac{\partial H}{\partial I}(I,\phi,t;\varepsilon) \\ \dot I &= -\frac{\partial H}{\partial \phi}(I,\phi,t;\varepsilon) \end{aligned}$$

To understand the dynamics we use the Poincaré map defined in $\Sigma_\theta \simeq \mathbb{R}\times\mathbb{T}$:

$$\begin{aligned} \mathcal{P}_\theta : \mathbb{R}\times\mathbb{T} &\to \mathbb{R}\times\mathbb{T} \\ (I,\phi) &\mapsto \Phi(\theta+2\pi;\theta,I,\phi;\varepsilon) \end{aligned}$$

The case of one and a half degrees of freedom

- For $\varepsilon = 0$, $H(I, \phi, s; 0) = H_0(I)$, the system is integrable.
- $\Phi(t; \theta, I_0, \phi; 0) = \varphi(t - \theta, I_0, \phi) = (I_0, \phi + \omega(I_0)(t - \theta))$, $\omega(I_0) = \nabla H_0(I_0)$.
- The Poincaré map $\mathcal{P}_\theta$ has 1-dimensional invariant tori (invariant curves)

$$\mathcal{T}_{I_0} = \{(I_0, \phi),\ \phi \in \mathbb{T}\} :$$

and $\mathcal{P}_\theta(I_0, \phi) = \varphi(2\pi, I_0, \phi) = (I_0, \phi + \omega_0(I)2\pi)$.

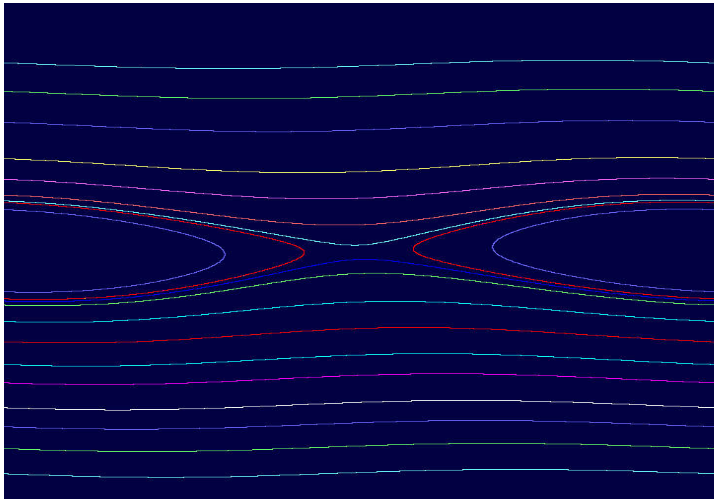
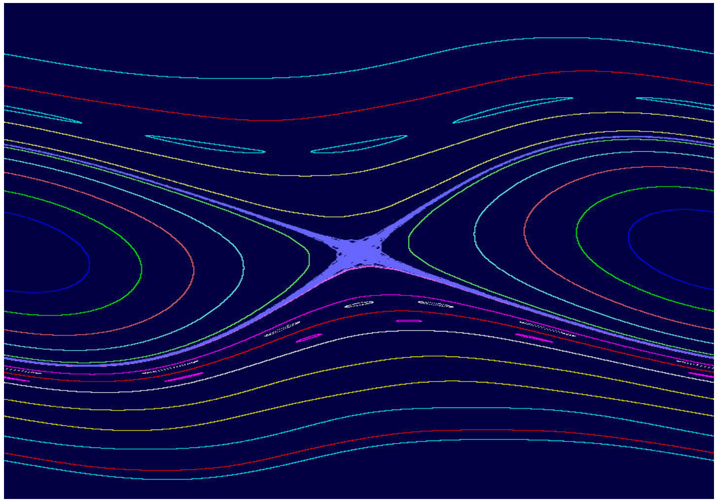
- Resonances $\omega(I_0) + \frac{l}{k} = 0$.

The case of one and a half degrees of freedom

- When $\varepsilon \neq 0$ the KAM theorem gives tori $\mathcal{T}_{I_0,\varepsilon}$ close to $\mathcal{T}_{I_0}$ for the Poincaré map associated to the perturbed Hamiltonian system.
- We know that the invariant tori cover the whole space $\mathbb{R} \times \mathbb{T}$ except a set of measure $\sqrt{\varepsilon}$.
- These tori (curves) are barriers to the unstable motion.

Answer to the Instability question: Instability (diffusion) is not possible.

The case of one and a half degrees of freedom

- Answer to the Instability question: No Instability, all the perturbed motions are close to the unperturbed ones.
- Near the resonant values the tori (curves in this case) have been destroyed and “new” objects appear.
- In the zones between the KAM curves we see “again” the invariant manifolds of a fixed point of the Poincaré map, but now the invariant manifolds do not coincide! This is known as the splitting of separatrices phenomenon. We need a method that “measures” this splitting.
- The system looks chaotic (local chaos)!

Splitting in one and a half degrees of freedom: the model

If we consider near integrable system

$$\mathcal{H}(I,\phi,t;\varepsilon)=\mathcal{H}_0(I)+\varepsilon\mathcal{H}_1(I,\phi,t)=\frac{I^2}{2}+\varepsilon\bar{V}(I,\phi)+\varepsilon\tilde{V}(I,\phi,t)$$

One can think on $\bar{V}(I,\phi)$ as the average part (respect to t) of the function $\mathcal{H}_1(I,\phi,t)$ and $\tilde{V}(I,\phi,t)$ as the rest:

$$\begin{aligned}\bar{V}(I,q) &= \int_0^{2\pi}\mathcal{H}_1(I,\phi,t)\,dt\\ \tilde{V}(I,\phi,t) &= \mathcal{H}_1(I,\phi,t)-\bar{V}(I,\phi)\end{aligned}$$

Splitting in one and a half degrees of freedom: the model

Exercice: If we now do the changes:

$$I = \sqrt{\varepsilon} p, \quad q = \phi$$

and the change of time: $\tau = \sqrt{\varepsilon} t$, we obtain a new Hamiltonian system:

$$q' = p + \frac{\partial H_1}{\partial p}(q, p, \omega\tau, \mu) \quad (3)$$

$$p' = -V'(q) - \frac{\partial H_1}{\partial q}(q, p, \omega\tau, \mu) \quad (4)$$

where:$\mu = \sqrt{\varepsilon}$, $\omega = \frac{1}{\sqrt{\varepsilon}}$ and:

$$V(q) = \bar{V}(0, q), \; H_1(q, p, t; \mu) = \tilde{V}(\mu p, q, t) + \bar{V}(\mu p, q) - \bar{V}(0, q)$$

Even if the perturbation is $\mathcal{O}(1)$ the efects of the large frequency make the term H_1 behave as a perturbation.

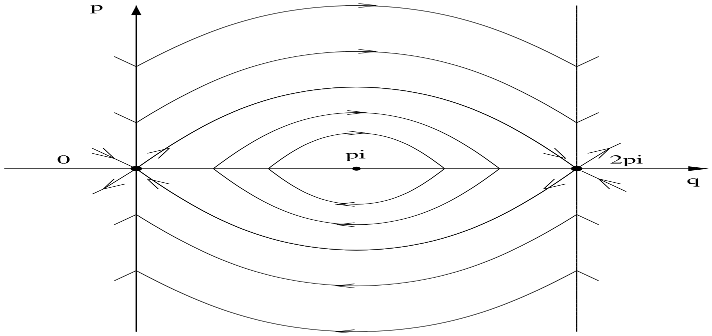
For this reason we will study systems of the form $\frac{p^2}{2} + V(q) + \mu H_1(p, q, \omega t)$.

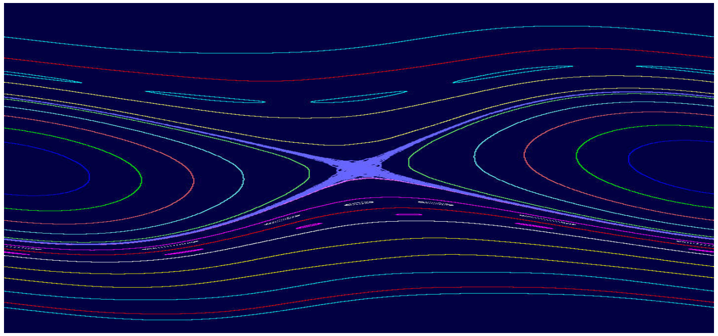
The case of one and a half degrees of freedom

We will study systems of the form $\frac{p^2}{2} + V(q) + \mu H_1(p, q, \omega t)$.
Remember that, in our setting: $\mu = \sqrt{\varepsilon}$. $\omega = \frac{1}{\sqrt{\varepsilon}}$.
To begin we will consider a Hamiltonian with $1 + \frac{1}{2}$ degrees of freedom with $2\pi-$periodic time depenence:

$$H(p, q, t; \mu) = H_0(p, q) + \mu H_1(p, q, t; \mu),$$

where $H_0(p, q) = \frac{1}{2}p^2 + V(q)$ is a pendulum, with $V(q)$ $2\pi-$periodic with a unique non-degenerate maximum, say at $q = 0$ and $V(0) = 0$.
For $\mu = 0$, $(0, 0)$ is and equilibrium of saddle type, with associated separatrices included in $H_0^{-1}(0)$.





We want to study what happends to the stable and unstable manifolds of $(0, 0)$ when $\mu \neq 0$.