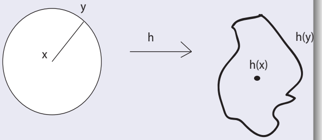
1D dynamics, Lecture 4:
.
Mating and conjugacy classes are real analytic manifolds

20 May 2024

Definition

An orientation preserving homeomorphism h is called *K-quasiconformal* if there exists a constant $K < \infty$ such that for each $x \in \mathbb{C}$

$$\liminf_{r \to 0} \frac{\sup_{|y-x|=r} |h(y) - h(x)|}{\inf_{|y-x|=r} |h(y) - h(x)|} \le K.$$



- If $K = 1$ then h is conformal (this is called Bers' Lemma).
- Such maps are, for example, Hölder and Lebesgue almost everywhere differentiable (as maps from $\mathbb{C} = \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$).
- (In general, a conjugacy h **cannot** be C^1, because then multipliers at corresponding periodic points would be equal: if $h \circ f \circ h^{-1} = g$ and $f^n(p) = p$ then $g^n(h(p)) = h(p)$ and from the chain rule $(f^n)'(p) = (g^n)'(h(p))$.

Beltrami coefficient

- H is orientation preserving.
- A quasiconformal map is almost everywhere differentiable.
- Its Beltrami coefficient is defined by

$$\mu_H(z) = \bar{\partial} H / \partial H$$

(write H is $z, \bar{z}$ coordinates).

- μ_H is a.e. defined, and $||\mu|| := \sup_z |\mu_H(z)| < 1$.
- If H is differentiable at z then H sends $DH(z)$ sends ellipses of eccentricity

$$\frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

to circles.

- Here the eccentricity is the ratio of the major to the minor axis of the ellipse.

Measurable Riemann Mapping Theorem

Theorem

- *For each measurable* $\mu\colon \bar{\mathbb{C}} \to \mathbb{D}$ *such that* $||\mu|| \le K < 1$ *there exists a unique* K*-quasiconformal map* $H = H_\mu$ *so that*

$$\mu_H = \mu \text{ a.e.}$$

  and so that $H(0) = 0, H(1) = 1, H(\infty) = \infty$.
- *If for each* $z \in \mathbb{C}$, *we have that* $\mu_t(z)$ *depends analytically on* $t \in C$ *then* $t \mapsto H_{\mu_t}(z)$ *is also analytic for each* $z \in \mathbb{C}$.

Theorem (Corollary)

Suppose that $f(z) = z^2 + a$ and $f(z) = z^2 + b$ both have finite critical orbits, and that they are conjugate. Then $a = b$.

Sketch proof:

- Let's take $a, b \in \mathbb{R}$ and take them so that $|a - b|$ is maximal.
- Construct qc conjugacy H between f_a and f_b, using Sullivan's pullback argument. See board.
- So $H^{-1} \circ f_a \circ H = f_b$. Let $\mu = \bar{\partial} H / \partial H$, and let $\mu_t = t\mu$.
- μ defines a measurable ellipse field which is invariant under f_a.
- Let H_t be the quasiconformal map associated to μ_t coming from the MRMT normalised $H_t(0) = 0$, $H_t(\infty) = \infty$ and $H_t(\mathbb{R}) = \mathbb{R}$.
- Since the ellipse field $t\mu$ is also invariant under f_a, we have that $g_t := H_t^{-1} \circ f_a \circ H_t$ is still an analytic map, with $g_1 = f_b$ and $g_0 = f_a$. So $g_t(z) = z^2 + c(t)$ with $c(t) \in \mathbb{R}\ \forall t \in [0, 1]$.
- Since $t \mapsto g_t$ is complex analytic on a neighbourhood of $[0, 1]$, we get a contradiction with the maximality of $|b - a|$.

Conjugacy classes of real analytic intervals maps form real analytic manifolds

- $\mathcal{T}_f^\nu = \{g \in \mathcal{A}^\nu; g$ topologically conjugate to f **and** all periodic points of g are hyperbolic $\}$.
- $\zeta(f) =$ maximal number of critical points *in the basins* of periodic attractors of f with pairwise disjoint infinite orbits.

Theorem (Trevor Clark & SvS)

1. *$\mathcal{T}_f^\nu$ is a real analytic manifold.*
2. *$\mathcal{T}_f^\nu \cap \mathcal{A}_a^\nu$ is a real analytic Banach manifold.*
3. *The codimension of $\mathcal{T}_f^\nu$ in the space of all real analytic functions is equal to $\nu - \zeta(f)$.*

Moreover, $\mathcal{T}_f^\nu$ is path connected. In fact, it's even contractible.

If there are periodic attractors without critical points in its basin we have to adjust this dimension.

* T. Clark and SvS, *Conjugacy classes of real analytic one-dimensional maps are analytic connected manifolds*, arXiv:2304.00883. Submitted for publication.

Remarks and strategy for proving Theorem B

Theorem (Trevor Clark & SvS)

1. *$\mathcal{T}_f^{\nu}$ is a real analytic manifold.*

Remarks.

- The orbits of the critical points of f are allowed to be infinite.
- **Avila, Lyubich and de Melo** proved this previously in the unimodal case, with a unique quadratic critical point.
- Their proof relies on the so-called Lambda lemma. (Note this is unrelated to the Lambda lemma from hyperbolic dynamics.)
- So need to assume $\exists$ at most one critical point .
- So we introduce rather different techniques: (i) **pruned polynomial-like maps**, (ii) **mating of such maps**, (iii) estimates for horizontal and vertical fields along a certain infinite dimensional set.

The approach by **Avila-Lyubich-de Melo (a sketch of the manifold statement)** in the **unimodal** case:

- Assume critical point is quadratic;
- From the Implicit Function Theorem, hybrid classes of hyperbolic maps and Misiurewicz (preperiodic) mappings are codimension-one analytic manifolds.
- Need to show they have uniform size. Then these manifolds are the leaves of a holomorphic motion in the parameter space.
- By the λ-Lemma (Bers-Royden, Sullivan-Thurston) the holomorphic motion extends to the closure of the set of hyperbolic mappings (so to the whole space, by density of hyperbolicity). The leaves through each point is an analytic manifold which coincides with the topological conjugacy class.

To apply the λ-Lemma the infinite dimensional manifolds need to be codimension-one, so **this argument breaks down for mappings with more than one critical point**.

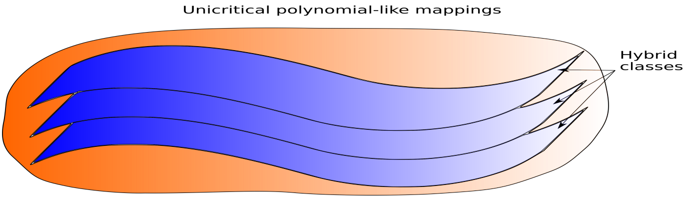
- To obtain manifold structure of $\mathcal{H}_f$ and $\mathcal{T}_f^{\nu}$ we use by techniques employed in the setting of quadratic-like mappings,
- i.e. use the mating construction of Lyubich (going back to work of Douady & Hubbard).
- However, our external maps have discontinuities and it is less clear that our space of external maps is a Banach manifold.
- we bypass this.

Inspiration: manifold structure for quadratic-like mappings

- Let $\mathcal{QL}$ be the space of real quadratic-like mappings, $f : U \to U'$ with $U \Subset U'$.
- Let $\mathcal{C} \subset \mathcal{QL}$ denote the set for which $K(f)$ is connected.
- Hybrid class $=$ Top class $+$ fixing multipliers at periodic attractors.

Theorem (Lyubich)

The hybrid class of $f \in \mathcal{C}$ is a connected, codimension-one, complex analytic submanifold of $\mathcal{QL}$.
Moreover, topological conjugacy classes laminate $\mathcal{C}$.



Hybrid conjugacy

We say that two polynomial-like maps $F\colon U \to U'$, $\tilde{F}\colon \tilde{U} \to \tilde{U}'$ are hybrid conjugate, if there exists a qc map $H\colon (U', U) \to (\tilde{U}', \tilde{U})$ so that $H \circ F = \tilde{F} \circ H$ and so that $\bar{\partial} H / \partial H = 0$ on $K(F)$.

Original approach for quadratic-like mappings

Remark: associated to each quadratic-like map there exists an expanding circle map $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$. Here expanding means $\exists k$ so that $|Dg^k(x)| > 1$ for each $x \in \partial\mathbb{D}$.

Theorem (Mating)

Let $F : U \to U'$ be a quadratic like map, and $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$ an expanding circle map. Then there exists a unique quadratic-like mapping $\widetilde{F} : \tilde{U} \to \tilde{U}'$, so that

1. *$\widetilde{F}$ is hybrid conjugate to F.*
2. *the external map of $\widetilde{F}$ is equal to g.*

Theorem

The space of expanding circle maps $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$ forms a Banach manifold.

Corollary: the space of expanding maps parametrises each hybrid conjugacy class.

Key idea in the proof of the Mating Theorem

- the expanding map $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$ extends to a degree two covering map $G\colon U_G \to U'_G$ where $U_G \Subset U'_G$.
- $F\colon U \setminus K(F) \to U' \setminus K(F)$ is also a degree two covering map.
- There exists a K-qc conjugacy H between $F\colon U \setminus K(F) \to U' \setminus K(F)$ and $G\colon U_G \setminus K(G) \to U'_G \setminus K(G)$. $H^{-1} \circ F \circ H = G$ on $U' \setminus K(G)$.
- Note that this map may not have a continuous extension.
- The Beltrami coefficient μ on $U' \setminus K(F)$ of H is F-invariant.
- Define

$$\nu = \begin{cases} \mu & \text{on } U' \setminus K(F) \\ 0 & \text{on } K(F) \end{cases}$$

- Now take H_ν to the qc map whose Beltrami coefficient is equal to ν and define

$$\tilde{F} = H_\nu^{-1} \circ F \circ H_\nu$$

  Since ν is F invariant, $\tilde{F}$ is conformal.
- This is the required map in the Mating Theorem.

Our approach is first to show that the Mating Theorem goes through in our context

Theorem (Mating)

Let $F : U_F \to U'_F$ and $G : U_G \to U'_G$ be pruned polynomial-like mappings with $Q(F) = Q(G)$. Then there exists a unique pruned polynomial-like mapping $\widetilde{F} : U_{\tilde{F}} \to U'_{\tilde{F}}$, so that

1. *$\widetilde{F}$ is hybrid conjugate to F.*
2. *$\widetilde{F}$ and G have the same external mappings.*

We call $\widetilde{F}$ a *mating* of F and g.

From this we obtain the theorem on the next slide.

Manifold structure of hybrid classes

Theorem

Let $F : U_F \to U'_F$ and $G : U_G \to U'_G$ be pruned polynomial-like mappings so that $Q(F) = Q(G)$. Then

- *the hybrid classes of F and G are homeomorphic;*
- *if the hybrid class of G has an analytic structure, then the one for F also has an analytic structure.*

Lemma

The hybrid class of any hyperbolic map forms a real analytic Banach manifold.

Corollary

The hybrid class of any interval map without parabolic periodic points is a real analytic Banach manifold.

What is needed in the previous slide is the following

Theorem

Two topologically conjugate maps in $\mathcal{A}^{\underline{\nu}}$ without parabolic periodic points are qs conjugate.

- T. Clark and SvS, *Quasisymmetric rigidity in one-dimensional dynamics*, https://arxiv.org/abs/1805.09284.

Tangent vectors to hybrid classes

Definition

E_f^h is the space of all holomorphic vector fields near I s.t. $\exists$ a pruned polynomial-like extension $F\colon U \to U'$ of f and a qc vector field α on U so that

$$v(z) = \alpha \circ F(z) - DF(z)\alpha(z) \text{ for } z \in U \tag{1}$$

and so that $\bar{\partial}\alpha = 0$ on K_F.

Proposition ($T_f\mathcal{H}_f = E_f^h$)

Given each $v \in E_f^h$ there exists a one-parameter family of maps $f_{t,v} \in \mathcal{H}_f$ with $f_{0,v} = f$, depending analytically on t and so that $\frac{d}{dt} f_{t,v}\big|_{t=0} = v$. Vice versa, for each $g \in \mathcal{H}_f$ near f there exists $v \in E_f^h$ so that $f_{1,v} = g$.

Codimension of these manifolds

To determine the codimension of the real analytic manifolds, we will use E_f^h and also a space E_f^u corresponding to the *vertical vectors*.

Important differences with ALM:

1. In ALM to each real analytic map they assign a puzzle map. The domain of this puzzle maps forms a necklace neighbourhood of the interval I (rather than an actual neighbourhood). To construct these puzzle maps, in ALM it is assumed that there are big bounds, and therefore that the map is unimodal and has a quadratic critical point. We do not (and cannot) assume that there are big bounds.
2. In ALM, vertical vectors (i.e. vectors transversal to the manifolds) are obtained, as in Kozlovski's PhD thesis, by constructing first smooth vertical vector fields and then using a polynomial approximation. This is one of the most subtle arguments in their paper, for which they use the above puzzle maps. Here we can avoid this discussion and argue as in the polynomial-like case in the spirit of Lyubich's hairiness paper.

Key Lemma

Theorem (Pullback argument)

Assume that $f, g \in \mathcal{A}_a^{\nu}$ do not have parabolic periodic points and are hybrid conjugate on I. Then the following holds.

- *If there exists a qc conjugacy between their pruned polynomial-like extensions $F\colon U_F \to U'_F$ and $G\colon U_G \to U'_G$ which has dilatation $\le \varkappa$ on $U'_F \setminus (U_F \cup \Gamma_F)$, $U_F \setminus \Gamma_F$ and $U_F \setminus (U'_F \setminus \Gamma_F)$ then F, G are hybrid-conjugate and $\varkappa$-qc conjugate on their entire domain.*

Proof:

- pullback argument
- quasisymmetric rigidity (this is due to Clark & SvS genereralising Kozlovski, Shen and SvS)
- absense of line fields on the Julia set.

Key Lemma

Theorem (qc bound)

For each $f \in \mathcal{A}_a^{\nu}$ so that all its periodic points are hyperbolic there exist $\delta > 0$ and $L > 0$ so that the following holds. Assume that $g_0, g_1 \in \mathcal{A}_a^{\nu}$ are real-hybrid conjugate to each other and that $||g_i - f||_\infty < \delta$. Then there exist pruned polynomial-like extensions $G_i \colon U_{G_i} \to U'_{G_i}$ of g_i, $i = 0, 1$ and a qc conjugacy h_{G_0,G_1} between them whose qc dilatation

$$\varkappa(h_{G_0,G_1}) \leq 1 + L||g_0 - g_1||_\infty.$$

Here $||\cdot||_\infty$ is the supremum norm on $\overline{\Omega_a}$.

Proof:

- previous theorem
- holomorphic motion.

Implication of Key Lemma: lower bound codimension

Assume that $f_n \in \mathcal{A}_a^\nu$ is critically finite and $f_n \to f$. Let $F_n \colon U_{F_n} \to U'_{F_n}$ and $F \colon U_F \to U'_F$ be pruned polynomial-like extensions. For any tangent vector field $v_n \in T_{f_n}\mathcal{H}_{f_n}$ there exists a qc vector field α_n so that

$$v_n(z) = \alpha_n(F_n(z)) - DF_n(z)\alpha_n(z).$$

The previous statement implies that there exists $C > 0$ so that

$$||\bar{\partial}\alpha_n||_{qc} \leq C||v_n||_{U_{F_n}}.$$

Corollary

- *Assume $v_n \in E^h_{f_n}$ is normalised and that v_n has a subsequence which converges to a vector v. Then $v \in E^h_f$.*
- *$codim(\mathcal{H}_f) \leq codim(\mathcal{H}_{f_n}) = \nu$.*

For vertical vector fields we also a similar estimate, using the pruned polynomial-like mapping structure, and this gives

$$\mathrm{codim}(\mathcal{H}_f) \geq \mathrm{codim}(\mathcal{H}_{f_n}) = \nu.$$