Lecture 3: Geometric methods in Arnold diffusion

Master Class
KTH, Stockholm

Tere M. Seara

Universitat Politecnica de Catalunya

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The case of two or more degrees of freedom

- In the case of one and a half degrees of freedom, the KAM tori (curves in $\mathbb{R}^2$) act as barriers to instability.
- Even if the separatrices of the periodic orbit (fix point for the Poincaré map) split, this only causes "local chaos" not global one.
- The previous argument does not work for periodic external perturbations of systems of two or more degrees of freedom (Poincaré maps of dimension 4 or higher!).
- $n = 2$: The KAM tori (2-dimensional) do not separate the phase space (4-dimensional) $(4 - 2 = 2 > 1)$.

The main conjecture:
"Typical systems in action-angle variables have orbits whose actions change widely **even if the systems are close to integrable**"
Arnold itself gave the most famous example, that we now explain.

Best known example in the mathematical literature: Arnol'd example:

$$H(I,\phi,t;\varepsilon,\mu) = H_0(I) + \varepsilon H_1(I;\phi,t;\varepsilon,\mu)$$
$$= \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)(\sin\phi_2 + \cos t),$$

For $\varepsilon = 0$ we have an integrable system $H = H_0(I) = \frac{1}{2}(I_1^2 + I_2^2)$, therefore

$$I(t) = I(0),\ \forall t \in \mathbb{R}$$

Theorem

For $0 < \mu \ll \varepsilon \ll 1$, there exist orbits of the Hamilton's equations with

$$|I(T) - I(0)| > 1\ .$$

Answer to the Instability question: Instability, there exit perturbed motions whose actions change $O(1)$ even if the perturbative parameter ε is small!

Geometric idea of the Arnol'd example

$$H(I,\phi,t;\varepsilon,\mu)=\frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\phi_1-1)+\mu\varepsilon(\cos\phi_1-1)\,(\sin\phi_2+\cos t)$$

- The phase space is 5 dimensional: $\mathbb{R}^2\times\mathbb{T}^3$.
- The Poincaré map is 4 dimensional: $\mathcal{P}_\theta:\Sigma_\theta\to\Sigma_\theta$, $\Sigma_\theta\simeq\mathbb{R}^2\times\mathbb{T}^2$
- $\varepsilon=0$: $H(I,\phi,t;0,0)=H_0(I)=\frac{1}{2}(I_1^2+I_2^2)$
- Equations: $\dot I_1=\dot I_2=0$, $\dot\phi_1=I_1$, $\dot\phi_2=I_2$
- Integrable system. Poincaré map $\mathcal{P}_\theta(x)=\varphi(2\pi;x)$ is given by:

$$\mathcal{P}_\theta(I_1^0,I_2^0,\phi_1^0,\phi_2^0)=(I_1^0,I_2^0,\phi_1^0+2\pi I_1^0,\phi_2^0+2\pi I_2^0)$$

- The 2- dimensional tori:

$$\mathbb{T}_{I^0}=\{I_1=I_1^0,\ I_2=I_2^0,\ (\phi_1,\phi_2)\in\mathbb{T}^2\}$$

  are invariant and foliate the space $\mathbb{R}^2\times\mathbb{T}^2$.
- The motion in the torus is quasiperiodic of frequency: $\omega(I^0)=(I_1^0,I_2^0)$.

Geometric idea of the Arnol'd example

$H(I,\phi,t;\varepsilon,\mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)\,(\sin\phi_2 + \cos t)$

- $\varepsilon > 0$, $\mu = 0$: **intermediate Hamiltonian:**

$$H(I,\phi,t;\varepsilon,0) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1)$$

- Integrable system (model of a simple resonance)

$$\begin{aligned} \dot\phi_1 &= I_1 \\ \dot I_1 &= \varepsilon \sin\phi_1 \\ \dot\phi_2 &= I_2 \\ \dot I_2 &= 0 \end{aligned}$$

- $I_2(t) = I_2(0)$, and $\phi_2(t) = \phi_2(0) + I_2(0)t$
- (I_1,ϕ_1) form a pendulum of Hamiltonian $P(I_1,\phi_1;\varepsilon) = \frac{1}{2}I_1^2 + \varepsilon(\cos\phi_1 - 1)$.

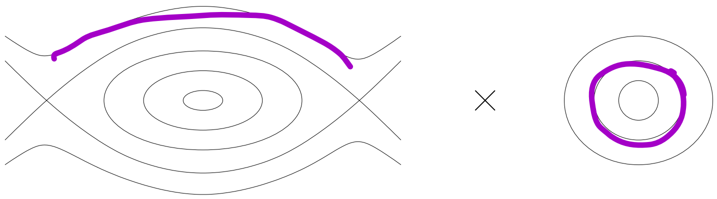
Geometric idea of the Arnol'd example

$$H(I,\phi,t;\varepsilon,\mu)=\tfrac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\phi_1-1)+\mu\varepsilon(\cos\phi_1-1)\,(\sin\phi_2+\cos t)$$

- $\varepsilon>0$, $\mu=0$: **intermediate Hamiltonian:**

$$H(I,\phi,t;\varepsilon,0)=\frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\phi_1-1)$$

- Integrable system (model of a simple resonance)
- Some 2 dimensional tori survive (KAM): they correspond to the rotational orbits in the pendulum.

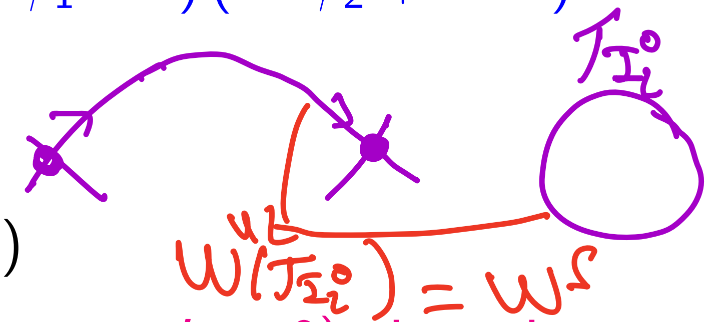


- 2-dimensional tori for the Poincaré map close to $I_1=I_1^0=\sqrt{2h}$. $I_2=I_2^0$, $h>1$:

$$\frac{1}{2}I_1^2+\varepsilon(\cos\phi_1-1)=h,\ \ h>1\quad I_2=I_2^0,\ \phi_2\in\mathbb{T}$$

Geometric idea of the Arnol'd example

$$H(I,\phi,t;\varepsilon,\mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$$



- $\varepsilon > 0$, $\mu = 0$: **intermediate Hamiltonian:** $H(I,\phi,t;\varepsilon,0) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1)$
- Integrable system (model of a simple resonance)
- Other tori are destroyed (correspond to the resonance $I_1 = 0$) given rise to whiskered one-dimensional tori.
- They are given by the critical point of the pendulum and the tori of the rotator, which become one-dimensonal tori for the Poincaré map of frequency $\omega = I_2$:

$$\mathcal{T}_{I_2^0} = \{I_1 = \phi_1 = 0,\ I_2 = I_2^0,\ \phi_2 \in \mathbb{T}\}, \quad \mathcal{P}_\theta(0,0,I_2^0,\phi_2) = (0,0,I_2^0,\phi_2 + 2\pi I_2^0)$$

- They are hyperbolic tori whose two-dimensional stable and unstable manifolds (whiskers) coincide along a homoclinic manifold.

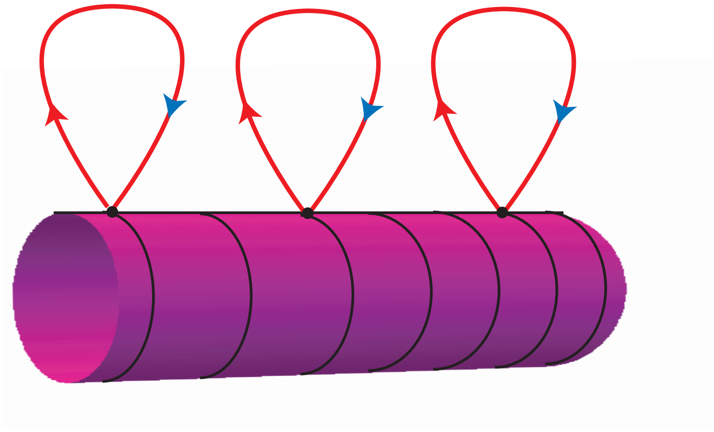
$$W^u(\mathcal{T}_{I_2^0}) = W^s(\mathcal{T}_{I_2^0}) = \{\frac{1}{2}I_1^2 + \varepsilon(\cos\phi_1 - 1) = 0, \quad I_2 = I_2^0,\ \phi_2 \in \mathbb{T}\}$$

Geometric idea of the Arnol'd example

$$H(I,\phi,t;\varepsilon,\mu)=\frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\phi_1-1)+\mu\varepsilon(\cos\phi_1-1)\,(\sin\phi_2+\cos t)$$

$\varepsilon>0$, $\mu=0$, **intermediate Hamiltonian:**

$$H(I,\phi,t;\varepsilon,0)=\frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\phi_1-1).$$



Dynamics of the intermediate Hamiltonian: $\varepsilon>0$, $\mu=0$

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

$$H(I,\phi,t;\varepsilon,\mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)\,(\sin\phi_2 + \cos t)$$

$$\begin{array}{rcl} \dot\phi_1 &=& I_1 \\ \dot I_1 &=& \varepsilon\sin\phi_1 + \varepsilon\mu\sin\phi_1(\cos t + \cos\phi_2) \\ \dot\phi_2 &=& I_2 \\ \dot I_2 &=& -\varepsilon\mu\cos\phi_2(\cos\phi_1 - 1) \end{array} \tag{1}$$

- For $\mu > 0$ all the 1-dimensional whiskered tori

$$\mathcal{T}_{I_2^0} = \{I_1 = \phi_1 = 0, \quad I_2 = I_2^0, (\phi_2, s) \in \mathbb{T}^2\}$$

  are preserved with the same dynamics: $\mathcal{P}_\theta(0,0,I_2^0,\phi_2) = (0,0,I_2^0,\phi_2 + 2\pi I_2^0)$.

- Each torus has 2-dimensional stable and unstable manifolds (whiskers) $W_\mu^{u,s}(\mathcal{T}_{I_2^0})$.

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

$H(I, \phi, t; \varepsilon, \mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$

For $\varepsilon > 0$, $\mu > 0$.

• For $\mu > 0$ the stable and unstable manifolds $W_\mu^{u,s}(\mathcal{T}_{I_2})$ (whiskers of $\mathcal{T}_{I_2}$) change.

• We need to prove that the 2-dimensional stable and unstable manifolds of the tori $\mathcal{T}_{I_2^0}$ intersect transversally along a homoclinic manifold (containing heteroclinic orbits bewteen the points of $\mathcal{T}_{I_2^0}$).

• This computation is the Poincaré-Melnikov method, analog to the one and a half degrees of freedom case for the computation of homoclinic intersections between the stable and unstable manifolds of periodic orbits.

• As we are in a 4-dimensional space for the Poincaré map, these transversal homoclinic intersections will give rise to heteroclinic ones between tori wich are close enough.

Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

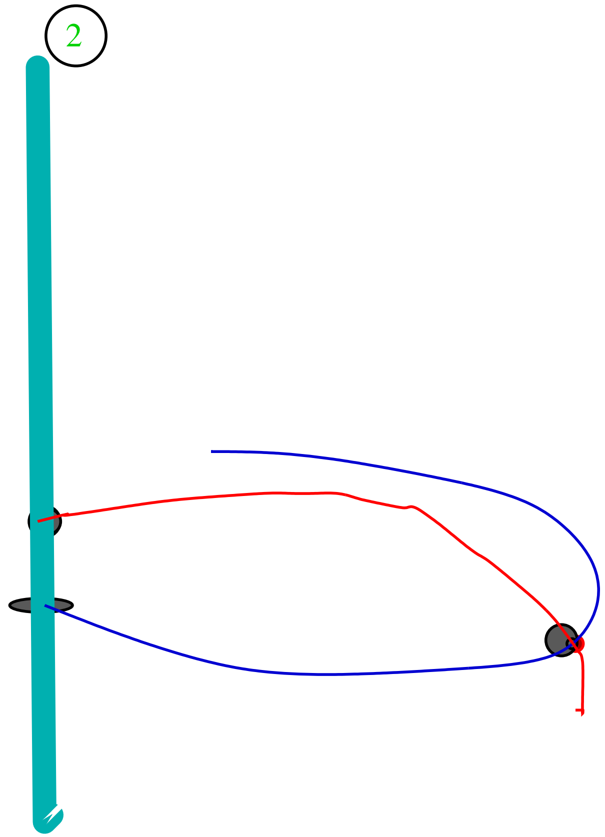
$H(I, \phi, t; \varepsilon, \mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$

- Scaling $I_1 = \sqrt{\varepsilon} p_1$, $I_2 = \sqrt{\varepsilon} p_2$ and $\tau = \sqrt{\varepsilon} t$:

$$\begin{aligned}
\dot{\phi}_1 &= p_1 \\
\dot{p}_1 &= \sin\phi_1 + \mu \sin\phi_1 (\sin\phi_2 + \cos\frac{\tau}{\sqrt{\varepsilon}}) \\
\dot{\phi}_2 &= p_2 \\
\dot{p}_2 &= -\mu \cos\phi_2 (\cos\phi_1 - 1)
\end{aligned}$$

- $K(p, \phi, \tau; \mu) = \frac{1}{2}(p_1^2 + p_2^2) + (\cos\phi_1 - 1) + \mu(\cos\phi_1 - 1)(\sin\phi_2 + \cos\frac{\tau}{\sqrt{\varepsilon}})$
- It is a $\mathcal{O}(\mu)$ perturbation of an integrable system with stable and unstable manifolds which coincide. So one can "generalize" the Melnikov approach.
- Warning! $\omega = \frac{1}{\sqrt{\varepsilon}}$, the Melnikov function will be exponentially small in $\sqrt{\varepsilon}$.
- If $\mu = \mathcal{O}(e^{-\frac{C}{\sqrt{\varepsilon}}})$, the calculation given by the Melnikov function is enough.
- The stable and unstable manifolds of every torus $\mathcal{T}_{I_2^0}$ intersect transversaly along a homoclinic orbit.

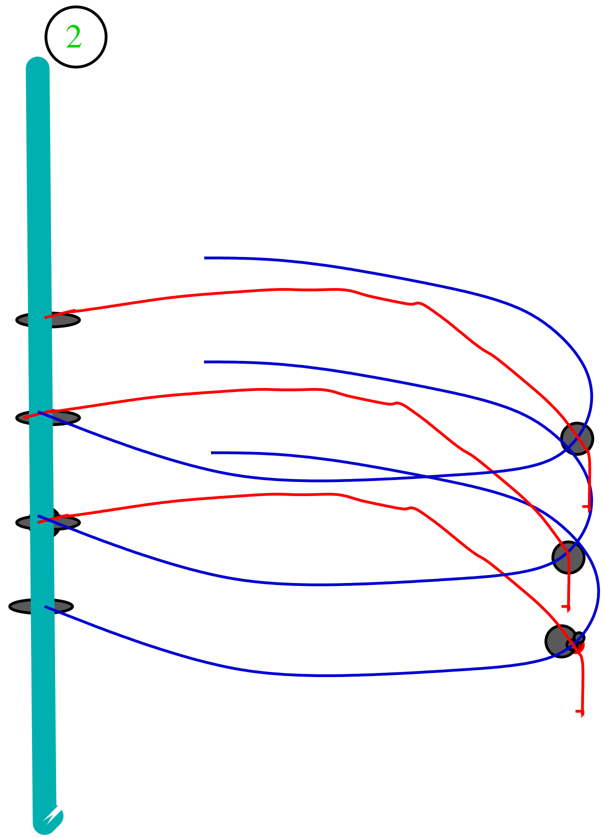
Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.



Transversal homoclinic orbits give rise to transversal heteroclinic orbits between tori $\mathcal{T}_{I_2^0}$ sufficiently close.
The unstable whisker of a torus $\mathcal{T}_{I_2^0}$ intersects transversally the stable whisker of another neighboring torus $\mathcal{T}_{I_2^1}$.

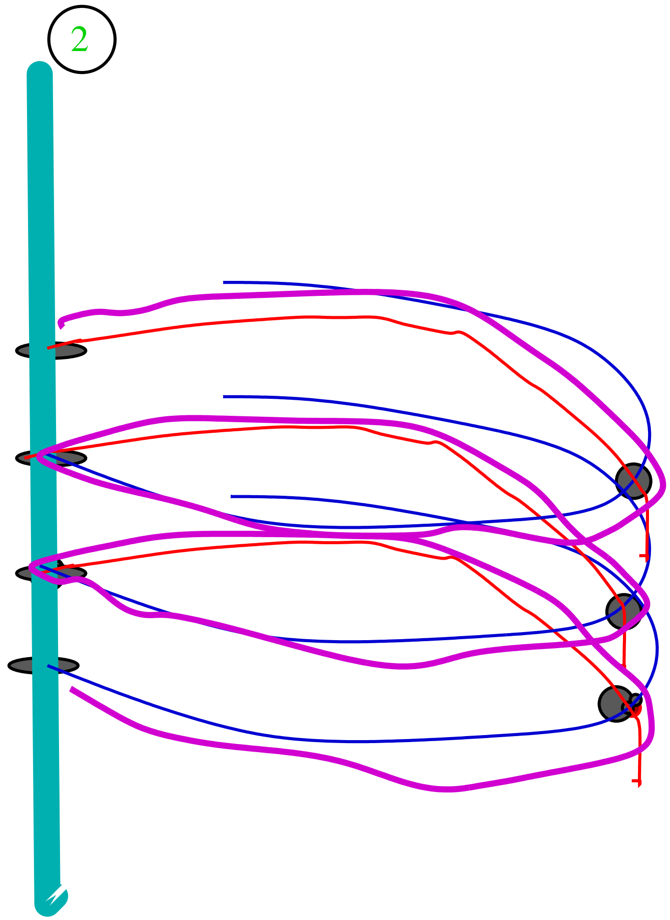
Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

We find $\{\mathcal{T}_{I_2^i}\}_{i=1}^N$ such that $W^u_{\mathcal{T}_{I_2^i}} \pitchfork W^s_{\mathcal{T}_{I_2^i}}$. (*transition chain*.)

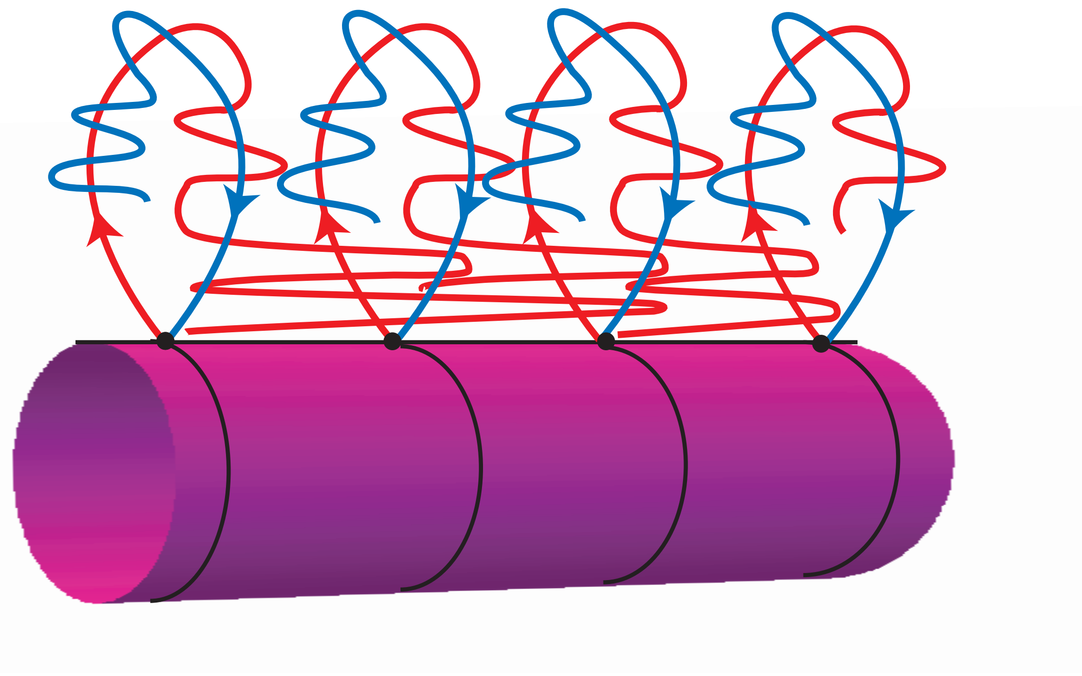


Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.

There is an orbit that shadows the transition chain. (*obstruction property*)



Diffusion mechanism when $\varepsilon > 0$, $\mu > 0$.



First observation: "a priori unstable systems"

The rigorous verification of Arnol'd mechanism uses the condition
$\mu = \mathcal{O}(e^{-\frac{C}{\sqrt{\varepsilon}}})$ (still open for $\mu = \mathcal{O}(\varepsilon^p)$)
P. Holmes, J. Marsden (1982): take the intermediate Hamiltonian as the unperturbed one: $\varepsilon = 1$, $0 < \mu \ll 1$

$$H(I, \phi, t; \mu) = \frac{1}{2}(I_1^2 + I_2^2) + \cos\phi_1 - 1 + \mu(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$$

Chierchia-Gallavotti:
a priori unstable system

$$H(I, \phi, t; \varepsilon) = \frac{1}{2}(I_1^2 + I_2^2) + \cos\phi_1 - 1 + \varepsilon h(I, \phi, t; \varepsilon)$$

a priori stable system

$$H(I, \phi, t; \varepsilon) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon h(I, \phi, t; \varepsilon)$$

Second observation: the "large gap" problem

Even in the a priori unstable system case, the Arnol'd example is based on the fact that all the 1-dimensional (hyperbolic) tori $\mathcal{T}_{I_2}$ are preserved.

- In general, $\mathcal{T}_{I_2}$ can be destroyed when $\varepsilon \neq 0$; KAM Theorem. Large gaps are typical.
- $\mathcal{T}_{I_2^0} = \{I_1 = \phi_1 = 0, \quad I_2 = I_2^0, \phi_2 \in \mathbb{T}\}$
  Motion on $\mathcal{T}_{I_2^0}$ is $\mathcal{P}_\theta(0, 0, I_2, \phi_2) = (0, 0, I_2, \phi_2 + I_2 2\pi)$ frequency $\omega = I_2$
- The gaps between the tori that survive are balls of radius $\sqrt{\varepsilon}$ centered in the resonances $(I_2 = m/n)$.
- The heteroclínic jumps are of order ε. (Melnikov theory would give $x^u - x^s = \varepsilon M(v, \phi, \theta) + O(\varepsilon^2)$).
- Arnold mechanism can not be applied to general perturbations of a priori unstable systems.

Geometric methods

- Arnold's mechanism is the begining of wat are called "geometric methods".
- But some new ideas came after his example.
- We will explain these methods and see how they apply to Arnold example and to other more general Hamiltonians.

The model:

$$H_\varepsilon(p,q,I,\phi,t) = \underbrace{h_0(I) + \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right)}_{H_0} + \varepsilon H_1(p,q,I,\phi,t;\varepsilon),$$

$$(p,q,I,\phi) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d$$

- Recall Arnold model:
  $H(I,\phi,t;\mu) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\phi_1 - 1) + \mu\varepsilon(\cos\phi_1 - 1)(\sin\phi_2 + \cos t)$
- Call $p = I_1$, and $q = \phi_1$, $\varepsilon = 1$ and $\mu = \varepsilon$ $I_2 = I$, $\phi_2 = \phi$ and
- $H(p,q,I,\phi,t;\mu) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + \cos q - 1 + \varepsilon(\cos q - 1)(\sin\phi + \cos t)$

First assumptions

$H_\varepsilon(p,q,I,\phi,t) = h_0(I) + \sum_{i=1}^n \pm\left(\frac{1}{2}p_i^2 + V_i(q_i)\right) + \varepsilon H_1(p,q,I,\phi,t;\varepsilon)$,

(A1.) The functions h_0, H_1 and V_i, $i = 1,\ldots,n$, are uniformly C^r for $r \geq r_0$.

(A2.) Each potential $V_i : \mathbb{T}^n \to \mathbb{R}$, $i = 1,\ldots,n$, is 2π-periodic in q_i and has a non-degenerate global maximum at 0, and hence each 'pendulum' $\pm\left(\frac{1}{2}p_i^2 + V_i(q_i)\right)$ has a homoclinic orbit to $(0,0)$, parametrized by $(p_i^0(\tau_i), q_i^0(\tau_i))$, $\tau_i \in \mathbb{R}$.

During this course we will take $n = d = 1$, and $h_0(I) = \frac{1}{2}I^2$, but the proofs can be easily generalized. The phase space for the flow will be 5-dimensional and for the Poincaré map $\mathcal{P}_{\theta,\varepsilon}$ will be 4-dimensional.

- $H_\varepsilon(p,q,I,\phi,t) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + V(q) + \varepsilon H_1(p,q,I,\phi,t;\varepsilon)$.

Main tools in geometric methods

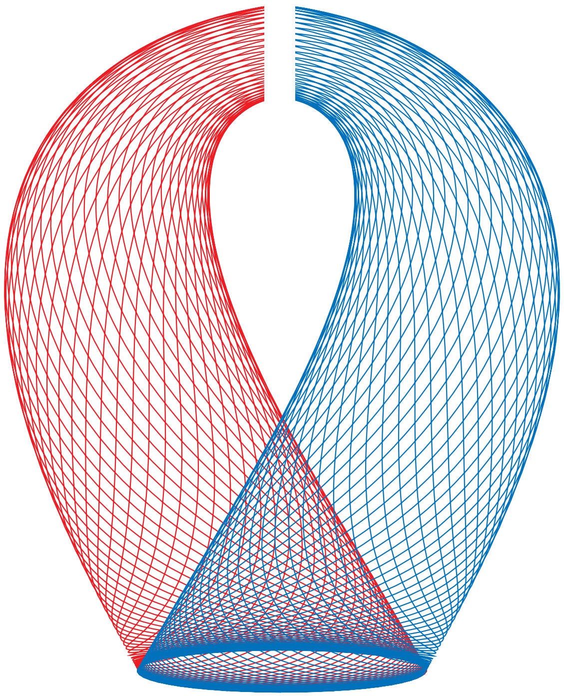
The main tools we will use are:

- Existence and persistence of normally hyperbolic invariant manifolds (NHIM)
- Existence and computation of the Scattering map in a NHIM

First tool: Normally hyperbolic invariant manifolds

Definition of a NHIM for a map (analogous for flows (Fenichel, Hirsch, Pugh, Shub, Pessin)):

- Normally hyperbolic invariant manifold (NHIM):
  - $F : M \to M$, C^r-smooth, $r \geq r_0$, $m = \mathrm{dim} M$.
  - $F(\Lambda) \subset \Lambda$, $n_c = \mathrm{dim}\Lambda$.
  - For any $x \in \Lambda$ we have.
    $T_x M = T_x\Lambda \oplus E_x^u \oplus E_x^s$
  - $n_s = \mathrm{dim} E^s$, $n_u = \mathrm{dim} E^u$.
  - $m = n_c + n_s + n_u$
  - $\exists\ C > 0$, $0 < \lambda < \mu^{-1} < 1$, s.t. $\forall\ x \in \Lambda$
    $v \in E_x^s \Leftrightarrow \|DF_x^k(v)\| \leq C\lambda^k \|v\|, \forall k \geq 0$
    $v \in E_x^u \Leftrightarrow \|DF_x^k(v)\| \leq C\lambda^{-k} \|v\|, \forall k \leq 0$
    $v \in T_x\Lambda \Leftrightarrow \|DF_x^k(v)\| \leq C\mu^{|k|} \|v\|, \forall k \in \mathbb{Z}$



Examples: hypebolic fix points, hypebolic periodic orbits.

First tool: Normally hyperbolic invariant manifolds

- The normal hyperbolicity of Λ implies that there exist smooth stable and unstable manifolds $W^{u,s}(\Lambda)$.
- If $x^{u,s} \in W^{u,s}(\Lambda)$ $\mathrm{dist}(F^n(x^{u,s}), \Lambda) \to 0$ as $n \to \mp\infty$.
- Moreover $W^{u,s}(\Lambda) = \bigcup_{x\in\Lambda} W^{u,s}(x)$ where
  $W^{u,s}(x) = \{x^{u,s},\ F^n(x^{u,s}) - F^n(x) \to 0,\ n \to \pm\infty\}$
- For any $x \in \Lambda$, $W^{u,s}(x)$ are smooth manifolds.
- In fact: $x^{u,s} \in W^{u,s}(x), \to\ \|F^n(x^{u,s}) - F^n(x)\| \leq K\lambda^{|n|},\ n \to \mp\infty$
- $W^{u,s}(x)$ are NOT invariant manifolds:

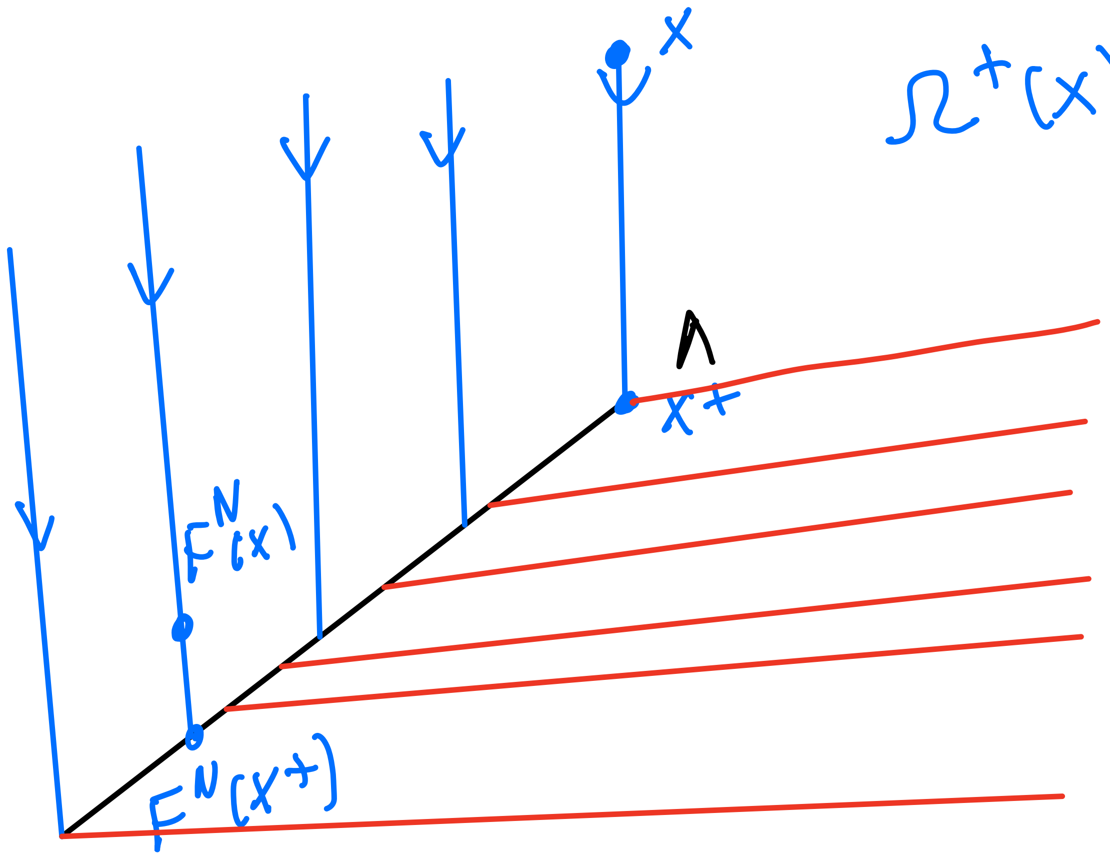
$$x^{u,s} \in W^{u,s}(x) \to F(x^{u,s}) \in W^{u,s}(F(x))$$

One can consider and the wave maps:

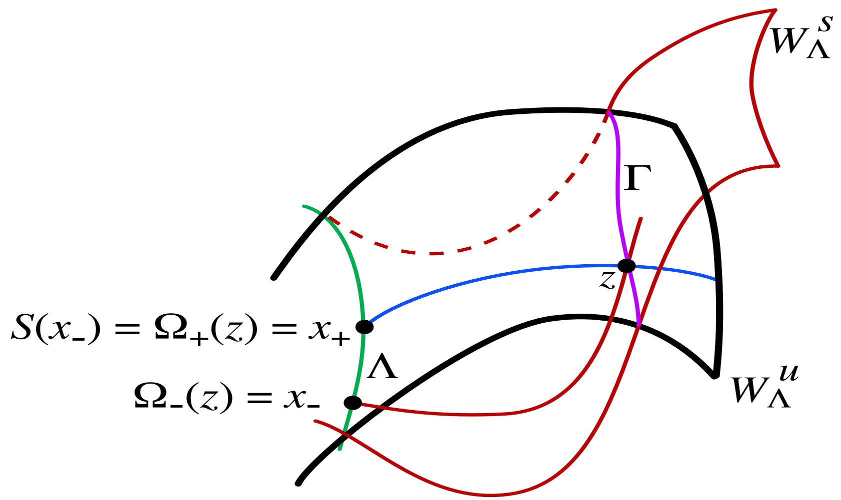
$$\begin{aligned}
\Omega^+ &: \quad W^s(\Lambda) \ni x \mapsto x_+ \in \Lambda,\ \text{such that } x \in W^s_{loc}(x_+)\\
\Omega^- &: \quad W^u(\Lambda) \ni x \mapsto x_- \in \Lambda,\ \text{such that } x \in W^s_{loc}(x_-)
\end{aligned}$$

These maps are smooth maps.

$\Omega^+(x) = x^+$



Second tool: The scattering map



- Assume that there exists a transverse homoclinic manifold $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$
- For each $x \in \Gamma$, we have

$$T_x M = T_x W^u(\Lambda) + T_x W^s(\Lambda), \quad T_x\Gamma = T_x W^u(\Lambda) \cap T_x W^s(\Lambda). \quad (2)$$

- For each $x \in \Gamma$, if $x_\pm \in \Lambda$ are such that $x \in W^s(x_+) \cap W^u(x_-)$. Then:

$$T_x W^s(\Lambda) = T_x W^s(x^+) \oplus T_x\Gamma, \quad T_x W^u(\Lambda) = T_x W^u(x^-) \oplus T_x\Gamma. \quad (3)$$

we say that Γ is a homoclinic channel.

Second tool: The scattering map

Scattering map associated to the homoclinic channel Γ.

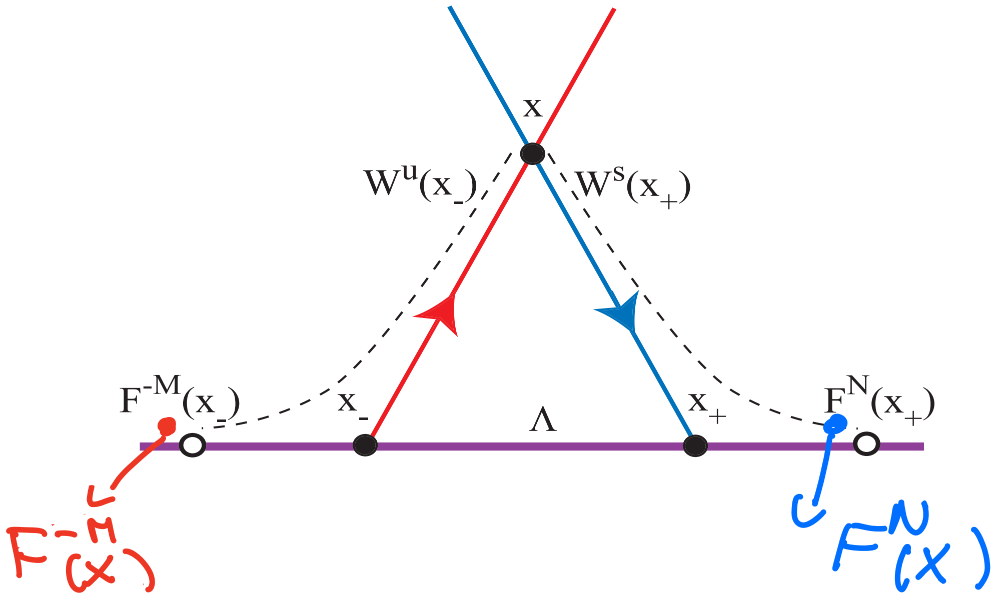
$$\sigma : \Omega^-(\Gamma) \subset \Lambda \to \Omega^+(\Gamma) \subset \Lambda, \quad \sigma = \Omega^+ \circ (\Omega^-)^{-1},$$

- It is a diffeomorphism from $\Omega^-(\Gamma)$ to $\Omega^+(\Gamma)$.
- If $\sigma(x^-) = x^+$, then there exits a unique $x \in \Gamma$ such that $W^u(x^-) \cap W^s(x^+) \cap \Gamma = \{x\}$.
- Note that:

$$\mathrm{dist}(F^{-n}(x) - F^{-n}(x_-)) \to 0, \quad \text{as } n \to \infty$$
$$\mathrm{dist}(F^m(x) - F^m(x_+)) \to 0 \quad \text{as } m \to \infty.$$

Second tool: The scattering map

The scattering map in Λ relates points x_- and $x_+ = \sigma(x_-)$ when there is an heteroclinic orbit between them.



- $F^{-M}(x)$ is close to $F^{-M}(x_-)$ and $F^N(x)$ is close to $F^N(x_-)$
- Call $x_1 = F^{-M}(x)$. Then x_1 is close to $F^{-M}(x_-)$, and $F^{N+M}(x_1)$ is close to $F^N(x_-)$
- **important:** There is no orbit from x_- to x_+ (it requires infinite time), but there is an orbit which begins close to x_- and arrives close to x_+.

The unperturbed problem ($\varepsilon = 0$): the NHIM

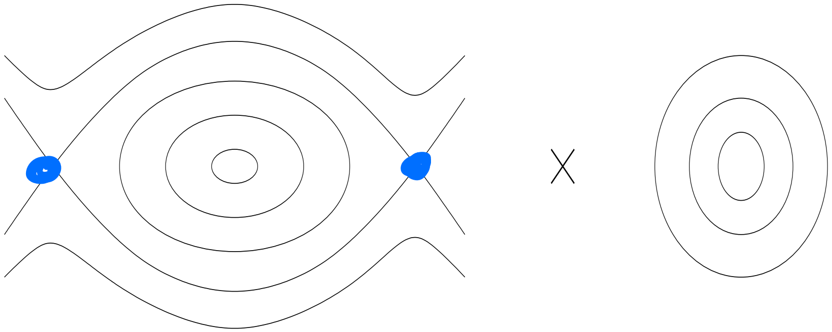
$H_\varepsilon(p, q, I, \phi, t) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + V(q) + \varepsilon H_1(p, q, I, \phi, t; \varepsilon)$

A different approach to Arnold diffusion: the use of normally hyperbolic manifolds.

When $\varepsilon = 0$, $H_0(p, q, I, \phi) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + V(q)$, the dynamics is:

$$
\begin{aligned}
\dot{q} &= p \\
\dot{p} &= V'(q) \\
\dot{\phi} &= I \\
\dot{I} &= 0
\end{aligned}
$$

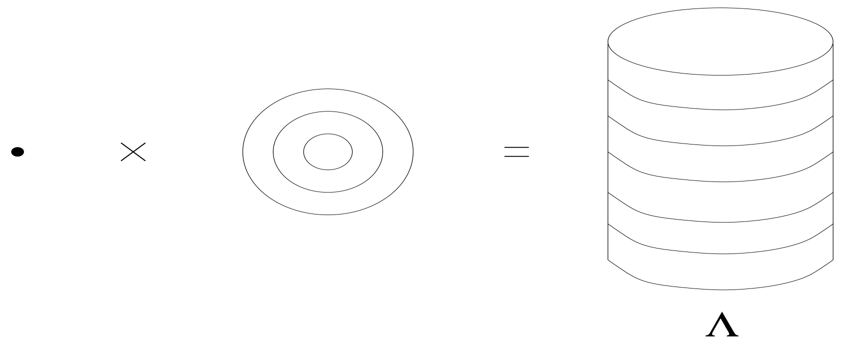
(p, q) form a pendulum, and (I, ϕ) a rotator: $I(t) = I^0$, $\phi(t) = \phi^0 + I^0 t$:



$p = q = 0$, $(I, \phi) \in \mathbb{R} \times \mathbb{T}$ is a 2-dimensional invariant manifold (cilinder) for the Poincaré map $\mathcal{P}_\theta$. $\mathcal{P}_\theta(0, 0, I, \phi) = (0, 0, I, \phi + 2\pi I)$

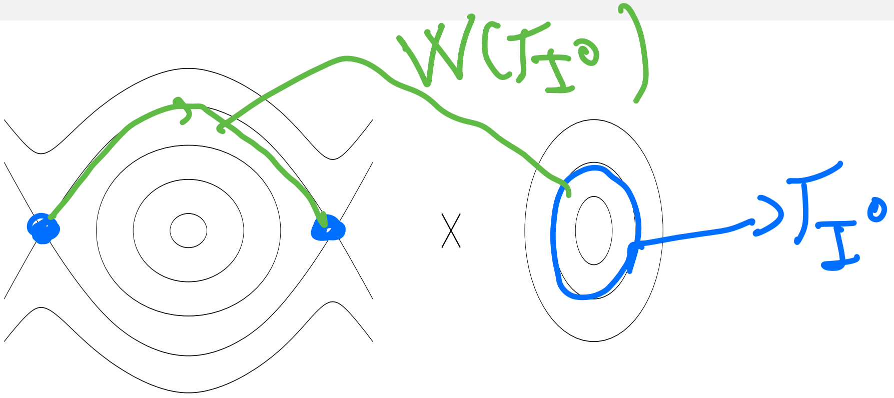
The unperturbed problem ($\varepsilon = 0$): the NHIM

$H_0(p, q, I, \phi) = \frac{1}{2} I^2 + \frac{1}{2} p^2 + V(q)$



- For any $I^0 \in \mathbb{R}$, $\mathcal{T}_{I^0} = \{(0, 0, I^0, \phi) : \ \phi \in \mathbb{T}\}$ is a 1-dimensional invariant torus with frequency $\omega(I^0) = I^0$.
- $\Lambda = \cup_{I \in \mathbb{R}} \mathcal{T}_I = \{(0, 0, I, \phi) : \ (I, \phi) \in \mathbb{R} \times \mathbb{T}\} \sim \mathbb{R} \times \mathbb{T}$
  is a 2-dimensional normally hyperbolic invariant manifold (cylinder) filled by 1-dimensional invariant tori $\mathcal{T}_I$.
- The Poincaré map $\mathcal{P}_\theta = \mathcal{P}_{\theta,0}$ restricted to Λ (inner motion) is given by $\mathcal{P}_\theta(0, 0, I, \phi) = (0, 0, I, \phi + 2\pi I)$
- (I, ϕ) are good global coordinates in Λ
- Λ and $\mathcal{T}_I$ are independent of θ.

The unperturbed problem ($\varepsilon = 0$): the stable and unstable manifolds of the NHIM



- Each torus $\mathcal{T}_{I^0}$ is a "whiskered torus" and its 2-dimensional stable and unstable manifolds coincide along a 2-dimensional homoclinic manifold: $W(\mathcal{T}_{I^0}) = \{(p, q, I^0, \phi),\ \frac{1}{2}p^2 + V(q) = 0,\ \phi \in \mathbb{T}\}$
- The 2-dimensional homoclinic manifold can be also parameterized by time: $W(\mathcal{T}_{I^0}) = \left\{ \left(p_h(v), q_h(v), I^0, \phi\right), v \in \mathbb{R}, \phi \in \mathbb{T} \right\}$ where $(p_h(v), q_h(v))$ is the homoclinic orbit of the pendulum: $(p_h(v), q_h(v)) \to 0$, as $v \to \pm\infty$
- Λ has 3-dimensional stable and unstable manifolds which coincide along the 3-dimensional homoclinic manifold given by the equation $\frac{1}{2}p^2 + V(q) = 0$, and can be parameterized by time; $\Gamma = \{(p_h(v), q_h(v), I, \phi), v \in \mathbb{R}^n, (I, \phi) \in \mathbb{R} \times \mathbb{T}\}$

The unperturbed problem ($\varepsilon = 0$): the Scattering map

$H_0(p, q, I, \phi) = \frac{1}{2}I^2 + \frac{1}{2}p + V(q)$

Introducing the parametrizations:

$$\begin{aligned} x_0 &= x_0(I, \phi) = (0, 0, I, \phi) \in \Lambda \\ x_h &= x_h(v, I, \phi) = (p_h(v), q_h(v), I, \phi) \in \Gamma \end{aligned}$$

the Poincaré map $\mathcal{P}_\theta$ acts, for any $\theta \in \mathbb{T}$, as

$$\begin{aligned} \mathcal{P}_\theta^n(x_0(I, \phi)) &= (0, 0, I, \phi + 2\pi I n) = x_0(I, \phi + 2\pi I n) \\ \mathcal{P}_\theta^n(x_h(v, I, \phi); 0) &= (\underbrace{p_h(v + 2\pi n), q_h(v + 2\pi n)}_{\downarrow\, n \to \pm\infty \atop 0}, I, \phi + I2\pi n) = x_h(v + 2\pi n, I, \phi + \end{aligned}$$

and it is therefore clear that $\forall v \in \mathbb{R}$ $\mathcal{P}_\theta^n(x_h; 0) - \mathcal{P}_\theta^n(x_0; 0) \xrightarrow[n \to \pm\infty]{} 0.$

That is, for any $v \in \mathbb{R}$: $x_h(v, I, \phi) \in W^s(x_0(I, \phi)) \cap W^u(x_0(I, \phi))$

$$\sigma_0(x_0) = x_0, \text{ in coordinates: } \sigma_0(I, \phi) = (I, \phi)$$

σ_0 is the identity on Λ.

The unperturbed problem ($\varepsilon = 0$): the Scattering map

$H_\varepsilon(p, q, I, \phi, t) = \frac{1}{2}I^2 + \frac{1}{2}p + V(q) + \varepsilon H_1(p, q, I, \phi, t; \varepsilon)$

When $\varepsilon = 0$ we have:

- The tori $\mathcal{T}_{I^0} = \{(0, 0, I^0, \phi) : \ \phi \in \mathbb{T}\}$ are invariant and foliate Λ.
- The scattering map $\sigma_0(I, \phi) = (I, \phi)$, which gives $\sigma_0 = Id$.
- In particular

$$\sigma_0(\mathcal{T}_I^0) = \mathcal{T}_I^0$$

- The unperturbed tori $\mathcal{T}_I^0$ only have homoclinic connexions.
- No possibility of diffusion
- Main idea in Arnold's proof:
  We want to see that, when $\varepsilon \neq 0$ we can define a scattering map such that, the image of one torus intersects other tori.

Sketch of the proof of Arnold diffusion using geometric methods:

1) Persistence of Λ.
2) Study of the inner dynamics on Λ_ε.
3) Study of stable and unstable manifolds for Λ_ε and their intersection: the Melnikov method.
4) The perturbative scattering map.
5) Transition chains.

(5') Combining the inner and the outer dynamics

6) Shadowing lemmas.

$\varepsilon \neq 0$, Step 1: persistence of Λ.

In the Arnold model:

$H(p,q,I,\phi,t;\varepsilon) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + \cos q - 1 + \varepsilon(\cos q - 1)(\sin\phi + \cos t)$

- $\Lambda = \{(0,0,I,\phi)\}$ persists for $\varepsilon > 0$ and the dynamics on it is unchanged. $\dot{I} = 0$, $\dot{\phi} = I$, $\mathcal{P}_{\theta,\varepsilon}(0,0,I,\phi) = (0,0,I,\phi + I2\pi)$.
- In particular, all the whiskered tori $\mathcal{T}_I$ are preserved for $\varepsilon > 0$.
- The manifold Λ has 3- dimensional stable and unstable manifolds, but these manifolds change. In particular they will not coincide anymore.
- To define a pertubed scattering map, we need to see that the invariant manifolds of Λ intersect transversally giving rise to a 2-dimensional homoclinic channel (to Λ) Γ_ε. This computation is the Poincaré, Melnikov method, analog to the one and a half degrees of freedom case.

$\varepsilon \neq 0$, Step 1: persistence of Λ.

$H_\varepsilon(p,q,I,\phi,t) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + V(q) + \varepsilon H_1(p,q,I,\phi,t;\varepsilon)$
As Λ is non compact, we restrict to $I \in [a,b]$, a compact interval the action space.
By the theory of NHIM applied to $\mathcal{P}_{\theta,\varepsilon}$, there exist smooth manifolds $\Lambda_{\theta,\varepsilon}$, $W^s_{loc}(\Lambda_{\theta,\varepsilon})$, $W^u_{loc}(\Lambda_{\theta,\varepsilon})$

$$\Lambda_{\theta,\varepsilon} = \Lambda + \mathcal{O}(\varepsilon),\ \ W^{s,u}(\Lambda_{\theta,\varepsilon}) = W^{s,u}(\Lambda) + \mathcal{O}(\varepsilon)$$

Moreover $W^{s,u}_{loc}(\Lambda_{\theta,\varepsilon}) = \bigcup_{x\in\Lambda_{\theta,\varepsilon}} W^{s,u}_{loc}(x)$.
That is, for any $x^{s,u} \in W^{s,u}_{loc}(\Lambda_{\theta,\varepsilon})$ there exist $x_\pm \in \Lambda_{\theta,\varepsilon}$ such that

$$\left|\mathcal{P}^n_{\theta,\varepsilon}(x^{s,u};\varepsilon) - \mathcal{P}^n_{\theta,\varepsilon}(x_\pm;\varepsilon)\right| \leqslant K\lambda_\varepsilon^{-|n|} n \to \pm\infty$$

The local manifolds can be globalized $W^{s,u}(\Lambda_{\theta,\varepsilon}) = \bigcup_{+,-n<0} \mathcal{P}^n_\theta(W^{s,u}_{loc}(\Lambda_{\theta,\varepsilon}))$.
The manifold $\Lambda_{\theta,\varepsilon}$ is not unique, not invariant, but only locally invariant. The local invariance means that there exists a neighborhood $\mathcal{V}$ of $\Lambda_{\theta,\varepsilon}$, such that any orbit of $\mathcal{P}_{\theta,\varepsilon}(x)$ that stays in $\mathcal{V}$ for all time is actually contained in $\Lambda_{\theta,\varepsilon}$.

$\varepsilon \neq 0$: Step 2: The Melnikov method

- In general, $W^s(\Lambda_{\theta,\varepsilon}) \neq W^u(\Lambda_{\theta,\varepsilon})$.
- To be able to define the scattering map in the perturbed case, we look for the points $x \in W^s(\Lambda_{\theta,\varepsilon}) \pitchfork W^u(\Lambda_{\theta,\varepsilon})$.
- Totally analogous to the one and a half degrees of freedom case we consider Poincaré function (or Melnikov potential) associated to the homoclinic manifold:

$$L(v, I, \phi, \theta) = -\int_{-\infty}^{\infty} \left[H_1(p_h(v+t), q_h(v+t), I, \phi + It, \theta + t; 0) - H_1(0, 0, I, \phi + It, \theta + t; 0)\right] dt.$$

$\varepsilon \neq 0$: Step 2: The Melnikov method

Proposition

Fix the section $\Sigma_\theta = \{(p, q, I, \phi, t),\ t = \theta\}$. *Assume that there exists a set* $U^- := \mathcal{I} \times \mathcal{J} \subset \mathbb{R}^\times\mathbb{T} \simeq \Sigma_\theta$, *such that* $\mathcal{I}$ *is a ball in* $\mathbb{R}$, *and for any values* $(I, \phi) \in U^-$, *the map*

$$v \in \mathbb{R}^n \to L(v, I, \phi, \theta) \in \mathbb{R}$$

has a non-degenerate critical point v^*, *which is locally given, by the implicit function theorem, by*

$$v^* = v^*(I, \phi, \theta).$$

Then, for $0 < |\varepsilon|$ *small enough, there exists a transversal homoclinic point* $x(I, \phi; \varepsilon) \in W^u(\Lambda_{\theta,\varepsilon}) \pitchfork W^s(\Lambda_{\theta,\varepsilon})$, *which is* ε*-close to the point* $x_h(v^*, I, \phi) = (p_h(v^*), q_h(v^*), I, \phi) \in \Gamma$*:*
that is:

$$x = x(I, \phi; \varepsilon) = (p_h(v^*) + \mathcal{O}(\varepsilon), q_h(v^*) + O(\varepsilon), I, \phi) \in W^s(\Lambda_{\theta,\varepsilon}) \pitchfork W^u(\Lambda_{\theta,\varepsilon}).$$

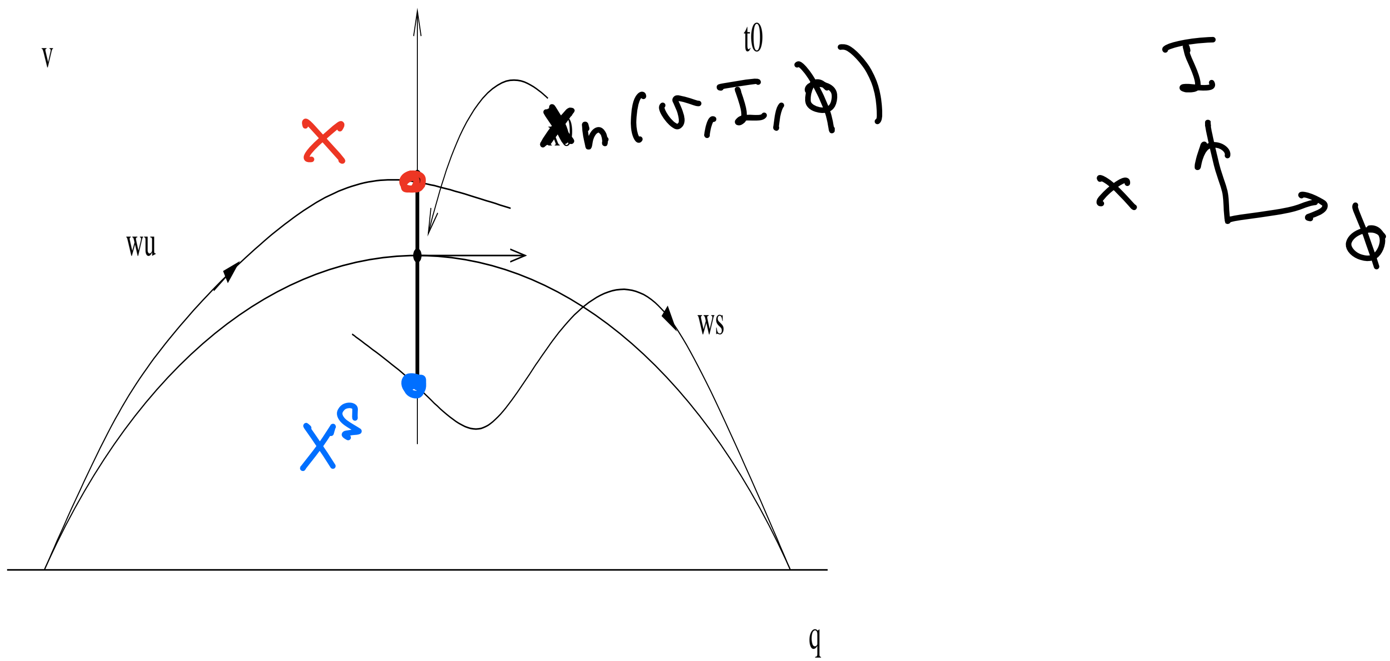
The proof is identical to the one and a half degrees of freedom.

$\varepsilon \neq 0$: Step 2: The Melnikov method

Proof:
Fix the Poincaré section Σ_θ and take any point

$$x_h = x_h(v, I, \phi) = (p_h(v), q_h(v), I, \phi) \in \Gamma$$

we have a straight line N transversal to Γ in x_h:



$N = N(x_h) = x_h + \langle \nabla P(p_h(v), q_h(v)) \rangle$, the normal bundle to the 3 separatrix Γ in the 4 dimensional space Σ_θ, where ∇P denotes the vector: $\nabla P = \left(\frac{\partial P}{\partial p}, \frac{\partial P}{\partial q}, 0, 0 \right)$.

$\varepsilon \neq 0$: Step 2: The Melnikov method

Since $W^{s,u}(\Lambda_{\theta,\varepsilon}) = W^{s,u}(\Lambda) + \mathcal{O}(\varepsilon)$, $W^{s,u}(\Lambda_{\theta,\varepsilon})$ intersect N in unique points $x^{s,u} \in W^{s,u}(\Lambda_{\theta,\varepsilon})$.

We try to find x_h and in particular v, such that $x^s = x^u$. Note that

$$x^{s,u} = \left(p_h(v) + \lambda^{s,u} \frac{\partial P}{\partial p}(p_h(v), q_h(v)), q_h(v) + \lambda^{s,u} \frac{\partial P}{\partial q}(p_h(v), q_h(v)), I, \phi \right),$$

where $x^{s,u} = x^{s,u}(v, I, \phi; \varepsilon)$ and $\lambda^{s,u} = \lambda^{s,u}(v, I, \phi; \varepsilon) = O(\varepsilon)$.

The computations done for one and half degrees of freedom give:

$$\begin{aligned} P(x^u) - P(x^s) &= \underbrace{P(x_-)}_{O(\varepsilon^2)} - \underbrace{P(x_+)}_{O(\varepsilon^2)} + \\ &+ \varepsilon \int_{-\infty}^{\infty} \{P, H_1\}(p_0(v+\sigma), q_0(v+\sigma), I, \phi + I\sigma, \theta + \sigma; 0) \\ &- \{P, H_1\}(0, 0, I, \phi + I\sigma, \theta + \sigma; 0) d\sigma \\ &+ \mathcal{O}(\varepsilon^2), \end{aligned}$$

where $x_{+,-} = x_0 + \mathcal{O}(\varepsilon) \in \Lambda_\varepsilon$ are the points such that $x^{u,s} \in W^{u,s}(x_\mp)$.

$\varepsilon \neq 0$: Step 2: The Melnikov method

Therefore

$$\begin{aligned} P(x^u) - P(x^s) &= \varepsilon \frac{\partial}{\partial v} \int_{-\infty}^{\infty} H_1(p_h(v+\sigma), q_h(v+\sigma), I, \phi + I\sigma, \theta + \sigma; 0) \\ &- H_1(0, 0, I, \phi + I\sigma, \theta + \sigma; 0) d\sigma + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon \frac{\partial}{\partial v} L(v, I, \phi, \theta) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

By the Implicit Function Theorem, non-degenerate critical points $v^* = v^*(I, \phi, \theta)$ of $v \in \mathbb{R} \mapsto L(v, I, \phi, \theta) \in \mathbb{R}$ give rise to $\tilde{v} = v^* + \mathcal{O}(\varepsilon)$ where $P(x^u) - P(x^s) = 0$ and, therefore, there are transversal homoclinic points

$$\begin{aligned} x^u &= x^s = x = x(I, \phi, \theta; \varepsilon) = \\ &= \left(p_h(\tilde{v}) + \lambda \frac{\partial P}{\partial p}(p_h(\tilde{v}), q_h(\tilde{v})), q_h(\tilde{v}) + \lambda \frac{\partial P}{\partial q}(p_h(\tilde{v}), q_h(\tilde{v})), I, \phi \right) \end{aligned}$$

in $W^s(\Lambda_\varepsilon) \pitchfork W^u(\Lambda_\varepsilon)$, where $\lambda = \lambda^{s,u} = \lambda(\tilde{v}, I, \phi, s; \varepsilon) = \mathcal{O}(\varepsilon)$, so that x is ε-close to $x_h(v^*, I, \phi) := (p_h(v^*), q_h(v^*), I, \phi)$, $v^* = v^*(I, \phi, \theta)$.

$\varepsilon \neq 0$: Step 2: The Melnikov method

- $L(v, I, \phi, \theta)$ is called the Melnikov potential.
- Once that we know that $x \in W^u(x_-) \cap W^s(x_+)$, where $x_\pm = (I_\pm, \phi_\pm) \in \Lambda_\varepsilon$ we want to compute the I coordinate of these points, that we already know $I_\pm = I + \mathcal{O}(\varepsilon)$.
- The same kind of computations using I instead of $P(p, q)$ give:

$$
\begin{aligned}
I(x^u) - I(x^s) &= I_- - I_+ + \\
&+ \varepsilon \int_{-\infty}^{\infty} \{I, H_1\}(p_h(v+\sigma), q_h(v+\sigma), I, \phi + I\sigma, \theta + \sigma) \\
&- \{I, H_1\}(0, 0, I, \phi + I\sigma, \theta + \sigma) d\sigma \\
&+ \mathcal{O}(\varepsilon^2) \\
&= I_- - I_+ + \varepsilon \frac{\partial}{\partial \phi} L(v, I, \phi\, s) + O(\varepsilon^2),
\end{aligned}
$$

Therefore, if we take $v = \tilde{v} = \tilde{v}(I, \phi, \theta; \varepsilon)$, then $x^u = x^s$ and: $I_- - I_+ = \varepsilon \frac{\partial}{\partial \phi} L(\tilde{v}, I, \phi, \theta) + \mathcal{O}(\varepsilon^2)$

$\varepsilon \neq 0$: Step 3: Formulas for the Scattering map

- We will now define the scattering map for the perturbed Hamiltonian.
- Take $\theta \in [0, 2\pi]$.
- Let $U^- := \mathcal{I} \times \mathcal{J} \subset \mathbb{R} \times \mathbb{T}$, such that $\mathcal{I}$ is a ball in $\mathbb{R}$, and for any values $(I, \phi, \theta) \in U^- \times [0, 2\pi]$ $\exists v^* = v^*(I, \phi, \theta)$ critical point of

$$v \mapsto L(v, I, \phi, \theta)$$

in such a way that

$$x = x(I, \phi, \theta; \varepsilon) \in W^s(\Lambda_{\theta,\varepsilon}) \pitchfork W^u(\Lambda_{\theta,\varepsilon}).$$

- Let $\Gamma_{\theta,\varepsilon} = \{x(I, \phi, \theta; \varepsilon), (I, \phi, \theta) \in U^- \times [0, 2\pi]\}$.
- For any $x \in \Gamma_{\theta,\varepsilon}$ there exist unique $x_\pm \in \Lambda_{\theta,\varepsilon}$ such that

$$\mathcal{P}^n_{\theta,\varepsilon}(x) - \mathcal{P}^n_{\theta,\varepsilon}(x_\pm) \xrightarrow[n\pm\infty]{} 0.$$

$\varepsilon \neq 0$: Step 3: Formulas for the Scattering map

Let

$$H_{\pm} = \bigcup \{x_{\pm}\} = \bigcup \{x_{\pm}(I, \phi, \theta; \varepsilon), (I, \phi, \theta) \in U^{-} \times [0, 2\pi]\}.$$

Then the scattering map associated to the homoclinic manifold $\Gamma_{\theta,\varepsilon}$ is $\sigma_{\theta,\varepsilon} : H_{-} \mapsto H_{+}$ such that $\sigma(x_{-}) = x_{+}$.

By the previous formula applied to $x^u = x^s = x = x(I, \phi, \theta; \varepsilon) \in \Gamma_{\theta,\varepsilon}$,

$$I_{+} - I_{-} = \varepsilon \frac{\partial}{\partial \phi} L(v^*, I, \phi, \theta) + \mathcal{O}(\varepsilon^2),$$

where $v^* = v^*(I, \phi, \theta)$. Calling,

$$L^*(I, \phi, \theta) = L(v^*, I, \phi, \theta)$$

we have that

$$I_{+} = I_{-} + \varepsilon \frac{\partial L^*}{\partial \phi}(I, \phi, \theta) + O(\varepsilon^2).$$

$\varepsilon \neq 0$: Step 3: Formulas for the Scattering map

It is easy to check that

$$L^*(I,\phi,\theta) = L^*(I,\phi - I\theta, 0) =: \mathcal{L}^*(I, \underbrace{\phi - I\theta}_{\alpha})$$

so that $L^*(I,\phi,\theta)$ depends essentially on two variables: I and $\alpha = \phi - I\theta$.
Therefore, defining the Poincaré reduced function as $\mathcal{L}^*(I,\alpha) = L^*(I,\alpha,0)$ we can write
$I_+ = I_- + \varepsilon\frac{\partial}{\partial\phi}\mathcal{L}^*(I,\alpha) + O(\varepsilon^2),\ \alpha = \phi - I\theta.$
Finally, by the geometric properties of the scattering map σ_ε is an (exact) symplectic and smooth map and therefore it satisfies:

$$\sigma_{\theta,\varepsilon}(I,\phi) = \left(I + \varepsilon\frac{\partial}{\partial\phi}\{\mathcal{L}^*(I,\alpha)\} + O(\varepsilon^2), \phi - \varepsilon\frac{\partial}{\partial I}\{\mathcal{L}^*(I,\alpha)\} + O(\varepsilon^2)\right),$$

where $\alpha = \phi - I\theta$, and $(I,\phi,\theta) \in U^- \times [0,2\pi]$.

$\varepsilon \neq 0$: Step 3: Formulas for the Scattering map

Summarizing:

$$\sigma_{\theta,\varepsilon}(I,\phi) = \left(I + \varepsilon \frac{\partial}{\partial \phi}\{\mathcal{L}^*(I,\alpha)\} + O(\varepsilon^2), \phi - \varepsilon \frac{\partial}{\partial I}\{\mathcal{L}^*(I,\alpha)\} + O(\varepsilon^2), s \right),$$

where $\alpha = \phi - I\theta$, and $(I,\phi,\theta) \in U^-$.
That is:

$$\sigma_\varepsilon = \mathrm{Id} - \varepsilon J \nabla \mathcal{L}^*(I,\alpha) + O(\varepsilon^2), \; J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore, except for an $\mathcal{O}(\varepsilon^2)$ error, $\sigma_{\theta,\varepsilon}$ is the ε-time map of the hamiltonian $-\mathcal{L}^*(I,\alpha)$, where $\alpha = \phi - I\theta$, and is therefore ε-close to the identity.

Computation of the Melnikov potential in the Arnold example

Remember the Arnold model after scaling variables and time:

$$H(p,q,I,\phi,t;\varepsilon,\mu)=\frac{1}{2}I^2+\frac{1}{2}p^2+(\cos q-1)+\mu(\cos q-1)\,(\sin\phi+\cos\frac{t}{\sqrt{\varepsilon}})$$

Perturbed parameter is μ, time frequency $\frac{1}{\sqrt{\varepsilon}}$.
The unperturbed system has $V(q)=\cos q-1$, the classical pendulum, and the homoclinic connection is

$$p_h(t)=\frac{2}{\cosh(t)},\quad q_h(t)=4\arctan e^t$$

and a perturbation H_1 of the form $H_1(p,q,I,\phi,t;\varepsilon)=(\cos q-1)g(\phi,r)$, $r=\frac{t}{\sqrt{\varepsilon}}$, where $g(\phi,r)=\sin\phi+\cos r$.

Computation of the Melnikov potential in the Arnold example

The Melnikov potetial satisfies: $L(v, I, \phi, \theta) = \mathcal{L}(I, \phi - Iv, \theta - v)$ where, using $\frac{p_h^2}{2} + \cos q_h - 1 = 0$ and that $p_h(t) = \frac{2}{\cosh t}$

$$
\begin{aligned}
\mathcal{L}(I, \phi, \theta) &= -\int_{-\infty}^{\infty} (H_1(p_h(t), q_h(t), I, \phi + It, \theta + t; 0) \\
&- -H_1(0, 0, I, \phi + It, \theta + t; 0))\, dt \\
&= -\int_{-\infty}^{\infty} (\cos q_h(t) - 1)\, g(\phi + It, \frac{\theta + t}{\sqrt{\varepsilon}})\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} p_h^2(t) \left(\sin(\phi + It) + \cos(\frac{\theta + t}{\sqrt{\varepsilon}})\right) dt \\
&= 2\int_{-\infty}^{\infty} \left(\sin\phi \cos(It) + \cos\frac{\theta}{\sqrt{\varepsilon}} \cos(\frac{t}{\sqrt{\varepsilon}})\right) \frac{1}{\cosh^2 t}\, dt \\
&= 2\sin\phi \int_{-\infty}^{\infty} \frac{\cos(It)}{\cosh^2 t}\, dt + 2\cos\frac{\theta}{\sqrt{\varepsilon}} \int_{-\infty}^{\infty} \frac{\cos(\frac{t}{\sqrt{\varepsilon}})}{\cosh^2 t}\, dt
\end{aligned}
$$

Computation of the Melnikov potential in the Arnold example

Using residues theorem, one can easily compute:

$$\int_{-\infty}^{\infty} \frac{\cos I\sigma}{\cosh^2 \sigma}\, d\sigma = \frac{\pi I}{\sinh \frac{\pi I}{2}}$$

And:

$$\int_{-\infty}^{\infty} \frac{\cos \frac{\sigma}{\sqrt{\varepsilon}}}{\cosh^2 \sigma}\, d\sigma = \frac{\pi}{\sqrt{\varepsilon}} \frac{1}{\sinh \frac{\pi}{2\sqrt{\varepsilon}}} = \frac{2\pi}{\sqrt{\varepsilon}} \frac{e^{-\frac{\pi}{2\sqrt{\varepsilon}}}}{1 - e^{-\frac{\pi}{\sqrt{\varepsilon}}}} \simeq \mathcal{O}(e^{-\frac{\pi}{2\sqrt{\varepsilon}}})$$

Computation of the Melnikov potential in the Arnold example

Therefore:

$$\mathcal{L}(I,\phi,\theta) = 2\pi\left(\sin\phi\frac{I}{\sinh\frac{\pi I}{2}} + \frac{1}{\sqrt{\varepsilon}}\cos\frac{\theta}{\sqrt{\varepsilon}}\frac{1}{\sinh\frac{\pi}{2\sqrt{\varepsilon}}}\right)$$

The critical points of $\mathcal{L}(I,\phi - Iv,\theta - v)$ can be computed and then the reduced Poincaré function $\mathcal{L}^*(I,\phi - I\theta)$ and the scattering map for this problem.

Remember that ($\alpha = \phi - I\theta$):

$$\sigma_\mu(I,\phi) = \left(I + \mu\frac{\partial}{\partial\phi}\{\mathcal{L}^*(I,\alpha)\} + O(\mu^2), \phi - \mu\frac{\partial}{\partial I}\{\mathcal{L}^*(I,\alpha)\} + O(\mu^2), s\right),$$

It is very easy to see that to obtain heteroclinic connections between two tori $I = I^0$ and $I = \bar{I}^0 > I^0$, we need that $\frac{\partial\mathcal{L}^*}{\partial\phi} > 0$!

That's a good exercice!

recall: to make everything rigourous we need $\mu \ll e^{-\frac{\pi}{2\sqrt{\varepsilon}}}$

Step 4: Studying the inner dynamics in $\Lambda_{\theta,\varepsilon}$

- Now we have a tool, the Scattering map, to "understand" the outer dynamics: the dynamics following the homoclinic excursions to $\Lambda_{\theta,\varepsilon}$.
- Next step is to understand the inner dynamics, that is, the dynamics in $\Lambda_{\theta,\varepsilon}$.
- Combining these two dynamics we will find "the skeleton" of the global ynamics.
- I will show you the classical methods that look for transition chains between the invariant objects inside $\Lambda_{\theta,\varepsilon}$.
- I will see that the classical KAM tori used by Arnold are not enough.
- I will show you a more recent result that does not need any knowledge about the invariant objects inside $\Lambda_{\theta,\varepsilon}$.

Step 4: Studying the inner dynamics in $\Lambda_{\theta,\varepsilon}$

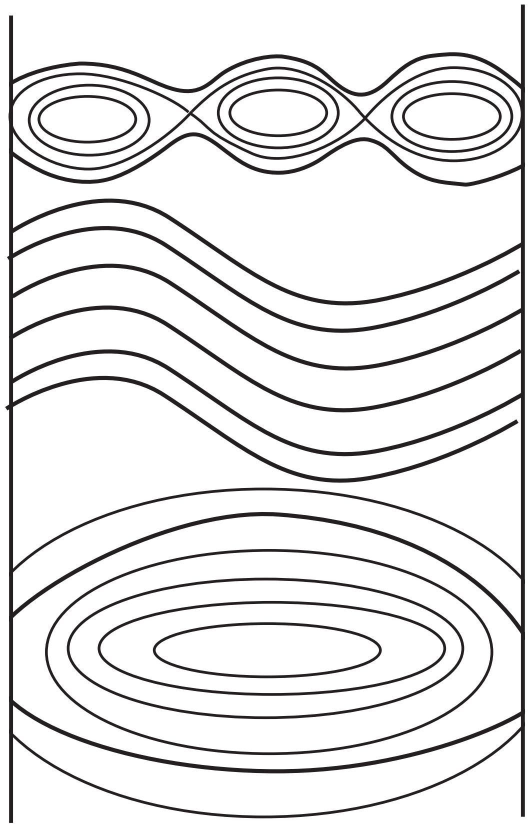
- The idea of some works using geometric methods is to find the invariant sets inside $\Lambda_{\theta,\varepsilon}$ which act as "barriers" to difuse along $\Lambda_{\theta,\varepsilon}$ and try to jump them throught $\sigma_{\theta,\varepsilon}$.
- More specifically, we follow Arnold's idea: we look for an collection of invariant tori $\mathcal{T}_i \subset \Lambda_{\theta,\varepsilon}$ such that the unstable manifold of $\mathcal{T}_i$ intersects trasversally the stable manifold of $\mathcal{T}_{i+1}$ giving a transition chain.
- We will describe the steps necessary to verify this mechanism.
- The verification uses standard methods from the geometric theory of perturbations but is long and technical:
  - 2.1 Compute the Hamiltonian flow in $\tilde{\Lambda}_\varepsilon = \cup_{\theta\in[0,2\pi]}\Lambda_{\theta,\varepsilon}$ and use the Averaging method to simplify the flow up to some order hight enough.
  - 2.2 Apply KAM theory to the averaged system. The whiskered tori and full dimensional tori inside $\Lambda_{\theta,\varepsilon}$.

Step 4.1: Averaging method

- As we have an 2-dimensional invariant manifold for any Poincaré section Σ_θ, we have a 3-dimensional invariant manifold for the flow, that we denote by $\tilde{\Lambda}_\varepsilon$.
- One can see that the reduced flow in $\tilde{\Lambda}_\varepsilon$ is hamiltonian and its a Hamiltonian is of the form: $\frac{1}{2}I^2 + \varepsilon K_1(I, \phi, t; \varepsilon)$ and can be computed at any order in ε.
- We look for a change of variables that reduce the system to motion of the actions to constant up to any order in ε (eliminates the angles ϕ, t from the hamiltonian)
- In one step of averaging the hamiltonian becomes: $\frac{1}{2}I^2 + \varepsilon h_1(I) + \varepsilon^2 K_1^1(I, \phi, t; \varepsilon)$.
- Averaging fails at resonances $Ik + l = 0$
- Far from resonances we obtain, after m steps $\frac{1}{2}I^2 + \varepsilon h_0(I; \varepsilon) + O(\varepsilon^m)$
- Close to simple resonances $I = (n_0/k_0)$ the motion is more complicated: $\frac{1}{2}I^2 + \varepsilon h_0(I; \varepsilon) + \varepsilon V(k_0\theta + n_0 t) + O(\varepsilon^m)$. Motion is pendulum like

Step 4.1: Averaging method

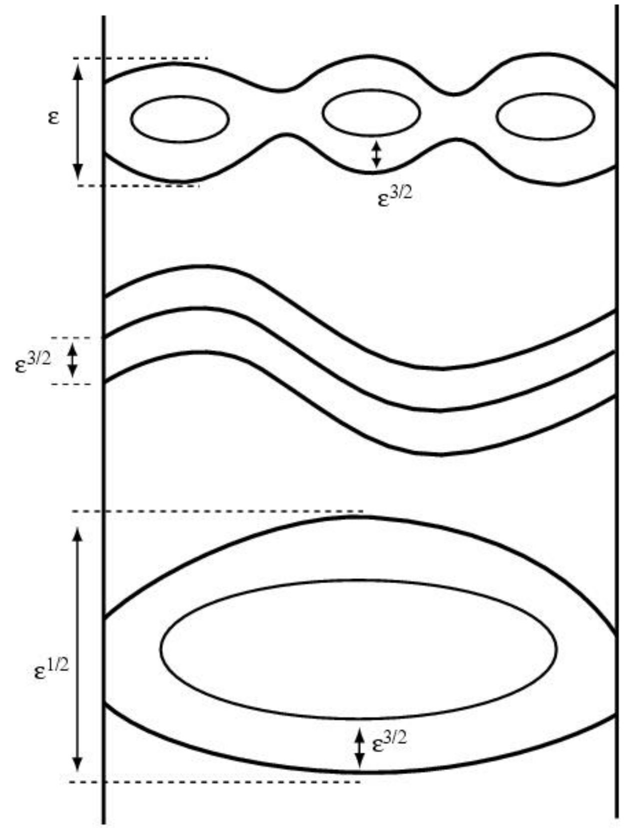
The motion of the Poincaré map for the averaged system in $\Lambda_{\theta,\varepsilon}$ is:



We see here the new objects that fill the gaps: the secondary tori!

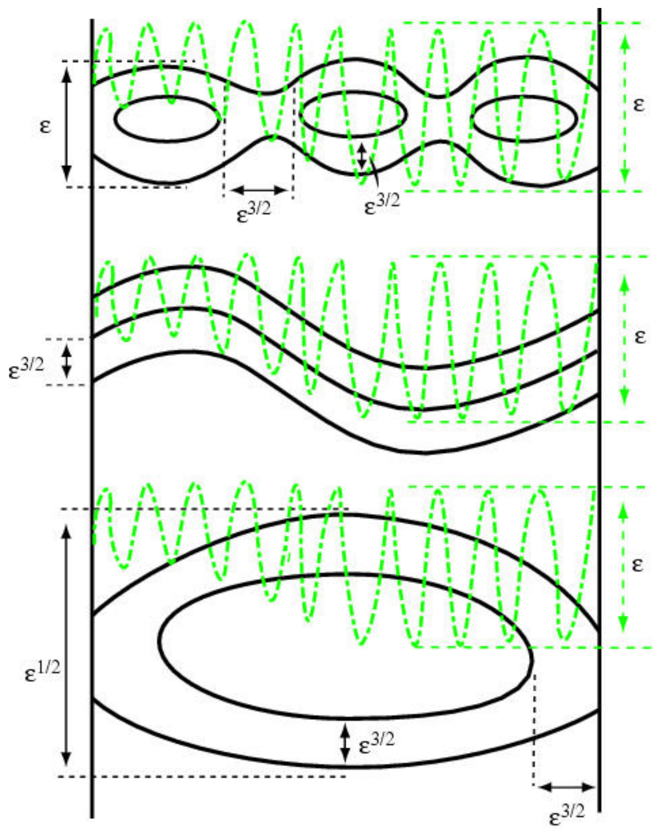
Step 4.2: KAM theorem

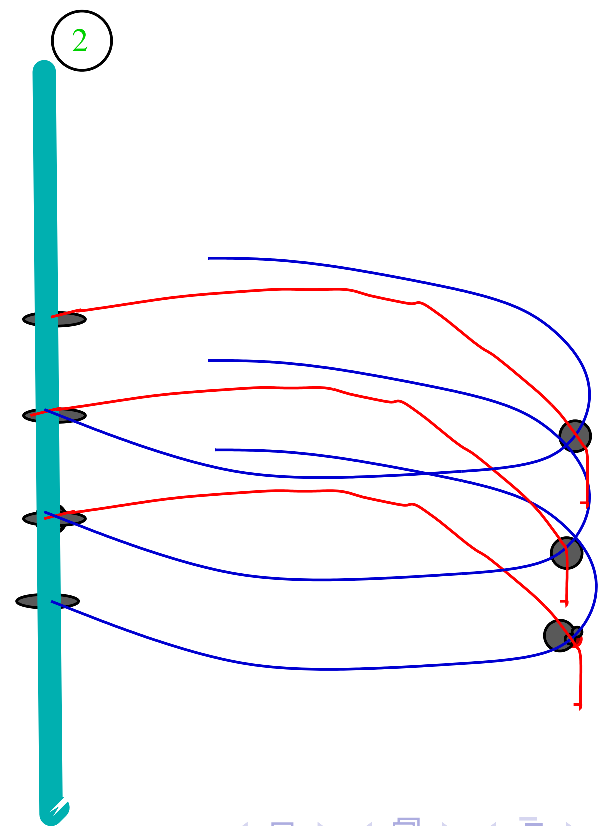
- Now we can apply the KAM theorem to the Poincaré map $\mathcal{P}_{\theta,\varepsilon}$ of the averaged system in $\Lambda_{\theta,\varepsilon}$ with $m = 3$.
- We obtain invariant tori (primary or secondary) at a distance $O(\epsilon^{3/2})$:



Step 5: Transition chains

- Once we know the structure in $\Lambda_{\theta,\varepsilon}$ given by the invariant tori of $\mathcal{P}_{\theta,\varepsilon}$ we use the scattering map $\sigma_{\theta,\varepsilon}$ to find heteroclinic intersections, even if the tori have different topology!.
- **Lemma:** If $\sigma_{\theta,\varepsilon}(\mathcal{T}_1) \cap \mathcal{T}_2 \neq \emptyset$ then $W^u(\mathcal{T}_1) \pitchfork W^s(\mathcal{T}_2)$; and therefore there is an heteroclinic connection between $\mathcal{T}_1$ and $\mathcal{T}_2$.





Step 6: Shadowing lemmas

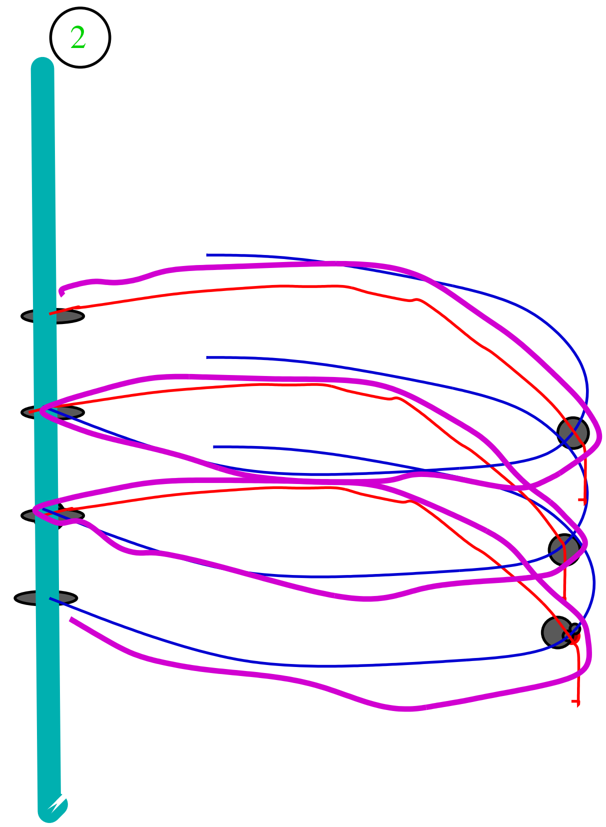
- We need to see that there is a "real" orbit which follows the chain.
- We use a Lambda lemma, Fontich-Martín (Nonlinearity, 2000) that can be applied to tori of different topology.
  - Let f be a symplectic map in a 4 symplectic manifold.
  - Assume that the map leaves invariant a $\mathcal{C}^1$ 1-dimensional torus $\mathcal{T}$ and that the motion in the torus is an irrational rotation.
  - Let ξ be a 2-dimensional manifold transversal to $W^u(\mathcal{T})$.

  Then, $W^s(\mathcal{T}) \subset \overline{\bigcup_{i>0} f^{-i}(\xi)}$.
- We use this lemma to see that:
  Let $\{\mathcal{T}_i\}_{i=1}^{\infty}$ be a sequence of transition tori (tori with irrational rotation, such that $W^u(\mathcal{T}_i) \pitchfork W^s(\mathcal{T}_{i+1})$)
  Given $\{\varepsilon_i\}_{i=1}^{\infty}$ a sequence of strictly positive numbers, we can find a point P and a increasing sequence of numbers T_i such that

$$\Phi_{T_i}(P) \in N_{\varepsilon_i}(\mathcal{T}_i)$$

  where $N_{\varepsilon_i}(\mathcal{T}_i)$ is a neighborhood of size ε_i of the torus $\mathcal{T}_i$.

Step 6: Shadowing lemmas



The orbit of P has action I which increase along the orbit!!

proof Let $x \in W^s_{\mathcal{T}_1}$. We can find a closed ballB_1, centered on x, and such that

$$\Phi_{T_1}(B_1) \subset N_{\varepsilon_1}(\mathcal{T}_1). \tag{4}$$

By the Lambda Lemma

$$W^s_{\mathcal{T}_2} \cap B_1 \neq \emptyset.$$

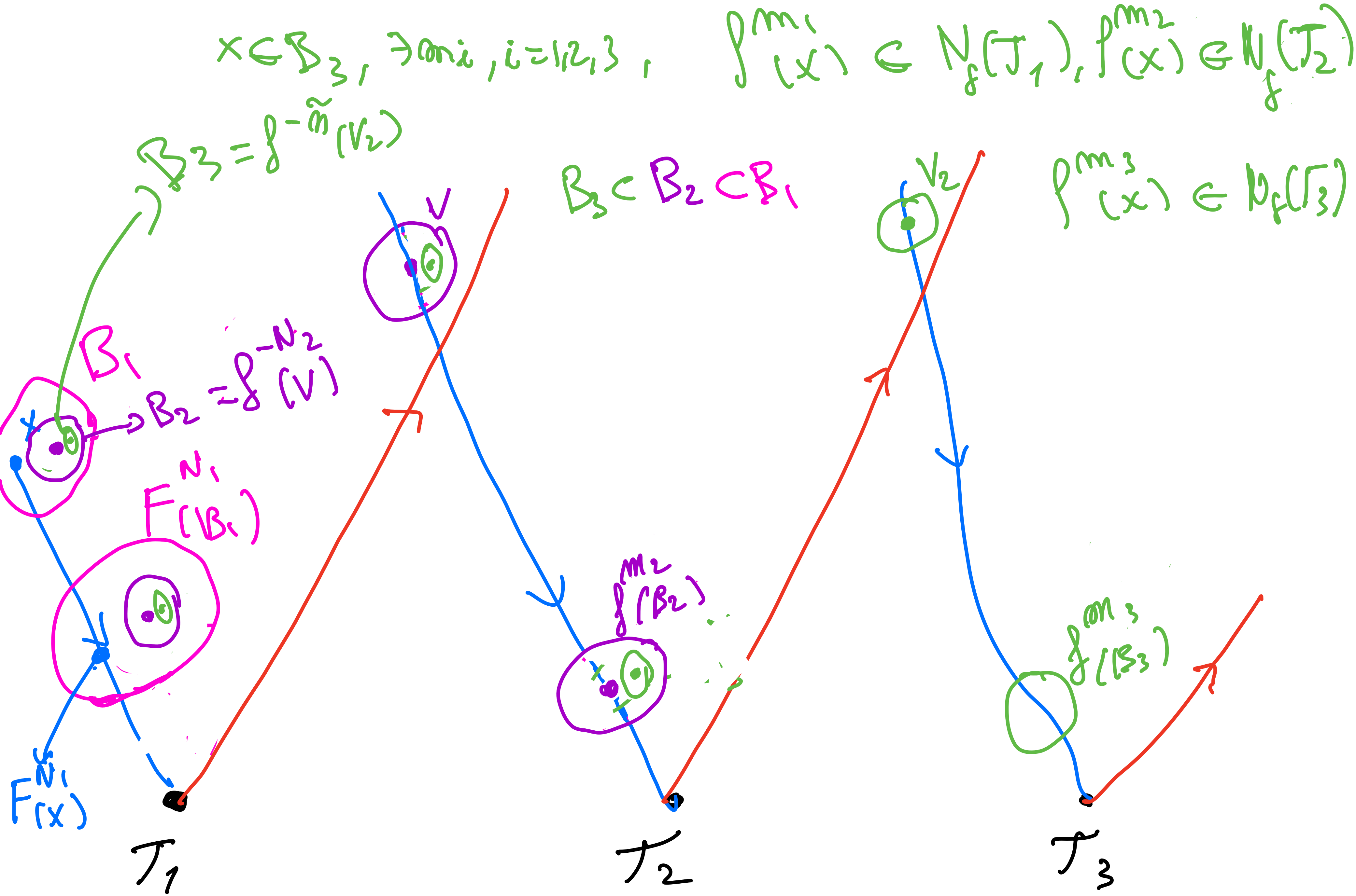
Hence, we can find a closed ball $B_2 \subset B_1$, centered in a point in $W^s_{\mathcal{T}_2}$ such that, besidessatisfying (4):

$$\Phi_{T_2}(B_2) \subset N_{\varepsilon_2}(\mathcal{T}_2).$$

Proceeding by induction, we can find a sequence of closed balls

$$B_i \subset B_{i-1} \subset \cdots \subset B_1$$
$$\Phi_{T_j}(B_i) \subset N_{\varepsilon_j}(\mathcal{T}_j), \quad i \leq j.$$

Since the balls are compact, $\cap B_i \neq \emptyset$.
A point P in the intersection satisfies the required property.


x ∈ B3, ∃ mi, i=1,2,3, f^m1(x) ∈ Nf(T1), f^m2(x) ∈ Nf(T2)
B3 = f^-m̃(V2)
B3 ⊂ B2 ⊂ B1
f^m3(x) ∈ Nf(T3)
V
V2
B1
B2 = f^-N2(V)
F^N1(B1)
F^N1(x)
f^m2(B2)
f^m3(B3)
T1
T2
T3